Inventory Model with Price-Dependent Demand Rate and No Shortages: An Interval-Valued Linear Fractional Programming Approach

Dr. Pavan Kumar

Assistant Professor, Department of Mathematics,
Faculty of Science and Technology,
IIS University, Jaipur-302020, India.

Abstract

In this paper, an interval-valued inventory optimization model is proposed. The model involves the price-dependent demand and no shortages. The input data for this model are not fixed, but vary in some real bounded intervals. The aim is to determine the optimal order quantity, maximizing the total profit and minimizing the holding cost subjecting to three constraints: budget constraint, space constraint, and budgetary constraint on ordering cost of each item. We apply the linear fractional programming approach based on interval numbers. To apply this approach, a linear fractional programming problem is modeled with interval type uncertainty. This problem is further converted to an optimization problem with interval-valued objective function having its bounds as linear fractional functions. Two numerical examples in crisp case and interval-valued case are solved to illustrate the proposed approach.

Keywords

Inventory model, Price-dependent demand, Interval-valued function, Fractional programming.

1. INTRODUCTION

For many practical inventory problems, some parameters are uncertain. Therefore to obtain more realistic results, the uncertainty in parameters must be considered, and the corresponding uncertain optimization methods must be constructed. Some techniques have been developed to solve uncertain optimization problems. The fuzzy and stochastic approaches are normally applied to describe imprecise characteristics. In these two types of techniques, the membership function and probability distribution play important roles.

This is, however, practically difficult to specify a precise probability distribution or membership function for some uncertain parameter. Over the past few decades, the interval number programming approach has been developed to deal with such type of problems in which the bounds of the uncertain parameters are only required. An optimization problem with interval coefficients is referred as the interval-valued optimization problem. In this case, the coefficients are assumed as closed intervals.

Charnes et al (1977) proposed a concept to solve the linear programming problems in which the constraints are considered as closed intervals. **Steuer** (1981) developed some algorithms to solve the linear programming problems with interval objective function coefficients. **Alefeld and Herzberger** (1983) introduced the concept of computations on interval-valued numbers. **Ishibuchi and Tanaka** (1990), and then **Chinneck and Ramadan** (2000) proposed the linear

DOI : 10.5121/oraj.2015.2402

programming problem (LPP) with interval coefficients in objective function. **Tong** (1994) studied the interval number and fuzzy number linear programming, and considered the case in which the coefficients of the objective function and constraints are all interval numbers. The possible interval-solution was obtained by taking the maximum value range and minimum value range inequalities as constraint conditions.

Moore and Lodwick (2003) further extended the theory of interval numbers and fuzzy numbers. **Mráz** (1998) proposed an approach to determine the exact bounds of optimal values in LP with interval coefficients. In the interval number programming, the objective function and constraints may not always be linear, but are often nonlinear. **Levin** (1999) introduced the nonlinear optimization under interval uncertainty. **Sengupta et al** (2001) studied the Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming.

The optimality conditions of Karush–Kuhn–Tucker (KKT) possess a significant role in optimization theory . In literature, various approaches to interval-valued optimization problems have been proposed in details, while few papers studied the KKT conditions for interval-valued optimization problems. **Wu** (2007) used the KKT conditions to solve an optimization problem with interval-valued objective function. **Wu** (2008a) studied the interval-valued nonlinear programming problems, and then **Wu** (2008b) & (2010) established the Wolfe duality for optimization problems with interval-valued objective functions. **Bhurjee and Panda** (2012) suggested a technique to obtain an efficient solution of interval optimization problem.

In single objective optimization, the aim is to find the best solution which corresponds to the minimum or maximum value of a single objective function. As far as the applications of nonlinear programming problems are considered, a ratio of two functions is to be maximized or minimized under certain number of constraints. In some other applications, the objective function involves more than one ratio of functions. The problems of optimizing one or more ratios of functions are referred as fractional programming problems. Normally, most of the multi objective fractional programming problems are first converted into single objective fractional programming problems and then solved. The decision maker (DM) must altogether optimize these conflicting goals in a framework of fuzzy aspiration levels. **Charnes and Cooper** (1962) proposed that a linear fractional program with one ratio can be reduced to a linear program using a nonlinear variable transformation. **Chanas and Kuchta** (1996) generalized the solution concepts of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals.

In the recent years, a kind of new applications of fractional programming was found in inventory optimization problems. **Sadjadi et al** (2005) proposed a fuzzy approach to solve a multi objective linear fractional inventory model. They considered two objective functions as fractional with two constraints: space capacity constraint, and budget constraint, and aimed to simultaneously maximize the profit ratio to holding cost and to minimize the back orders ratio to total ordered quantities. **Carla Oliveira et al**_(2007) presented an overview of multiple objective linear programming models with interval coefficients. Some applications of multi objective linear fractional programming in inventory were proposed by **Toksarı** (2008). **Effati et al** (2012) presented an approach to solve the interval-valued linear fractional programming problem.

Liu (2006) derived a computational method for the profit bounds of inventory model with interval demand and unit cost. **Liu and Wang** (2007) developed a method for numerical solution to interval quadratic programming. **Liu** (2008) discussed the posynomial geometric programming with interval exponents and coefficients.

Mishra (2007) studied some problems on approximations of functions in Banach Spaces, and then **Deepmala** (2014) studied the fixed point theorems for nonlinear contractions and its applications. Recently, **Vandana and Sharma** (2015) proposed an EPQ inventory model for non-instantaneous deteriorating item under trade credit policy. The concept of fuzzy was also used in case of fractional inventory model. **Kumar and Dutta** (2015) proposed a multi-objective linear fractional inventory model of multi-products with price-dependant demand rate in fuzzy environment.

This paper aims at extending the application of linear fractional programming to inventory optimization problems in interval uncertainty. We consider an interval-valued fractional objective function and interval-valued constraints for the proposed inventory problem. The paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, the notations and assumptions are given. In Section 4, a linear fractional programming inventory problem is proposed in crisp sense. In Section 5, we formulate above problem considering interval uncertainty. Two numerical examples are solved in Section 6. Finally, we conclude in Section 7.

2. PRELIMINARIES

Interval-Valued Number (Moore & Lodwick (2003))**:**

Let $* \in \{+, -, ., /\}$ be a binary operation on real line $\Re$. Let A and B be two closed intervals. Then, a binary operation is defined as

$$A * B = \{a * b : a \in A, b \in B\}$$

In the case of division, it is assumed that $0 \notin B$. The operations on intervals used in this paper are defined as:

a) $$kA = \begin{cases} [ka_L, ka_R]\,; & if\ k \geq 0 \\ (ka_R, ka_L)\,; & if\ k < 0 \end{cases} \quad (1)$$

b) $$A + B = [a_L\,,\, a_R] + [b_L\,,\, b_R] = [a_L + b_L, a_R + b_R] \quad (2)$$

c) $$A - B = [a_L\,,\, a_R] - [b_L\,,\, b_R] = [a_L\,,\, a_R] + [-b_R\,, -b_L] = [a_L - b_R, a_R - b_L] \quad (3)$$

d) $$A \times B = [\min\{a_Lb_L, a_Lb_R, a_Rb_L,\ a_Rb_R\},\ \max\{a_Lb_L, a_Lb_R, a_Rb_L,\ a_Rb_R\}] \quad (4)$$

e) $\frac{A}{B} = A \times \frac{1}{B} = [a_L\,,\, a_R] \times [\frac{1}{b_R}, \frac{1}{b_L}]$, provided $0 \notin [b_L\,,\, b_R]$,

Alternatively, division of interval numbers can be defined as

$$\frac{A}{B} = \frac{[a_L\,,\, a_R]}{[b_L\,,\, b_R]} = [\min\{\frac{a_L}{b_L}, \frac{a_L}{b_R}, \frac{a_R}{b_L}, \frac{a_R}{b_R}\}, \max\{\frac{a_L}{b_L}, \frac{a_L}{b_R}, \frac{a_R}{b_L}, \frac{a_R}{b_R}\}], \quad (5)$$

Alternatively, division of interval numbers can be defined as

$$\frac{A}{B} = \frac{[a_L\,,\, a_R]}{[b_L\,,\, b_R]} = [\frac{a_L}{b_R}, \frac{a_R}{b_L}], \quad \text{where } 0 \notin B,\ 0 \leq a_L \leq a_R, 0 < b_L \leq b_R. \quad (6)$$

f) $$A^k = [a_L, a_R]^k = \begin{cases} [1, 1]\ ; & if\ \ k = 0 \\ [a_L^k,\ a_R^k]; & if\ a_L \geq 0 \text{ or k odd} \\ [a_R^k,\ a_L^k]; & if\ a_R \leq 0 \text{ or k even} \\ [0, \max(a_L^k, a_R^k)];\ \text{if } a_L \leq 0 \leq a_R, & \text{and } k(>0) \text{is even} \end{cases} \quad (7)$$

Interval-Valued Function: Let $\Re^n$be an n-dimensional Euclidean space. Then a function

$$f: \Re^n \to I$$

is called an interval valued function (because $f(x)$ for each x∈ $\Re^n$ is a closed interval in $\Re$). Similar to interval notation, we denote the interval-valued function f with

$$f(x) = [f^L(x), f^U(x)],$$

where for each x∈ $\Re^n$, $f^L(x)$, $f^U(x)$ are real valued functions defined on $\Re^n$, called the lower and upper bounds of $f(x)$, and satisfy the condition:

$$f^L(x) \leq f^U(x).$$

Proposition 1. Let f be an interval valued function defined on $\Re^n$. Then f is continuous at c ∈ $\Re^n$ if and only if $f^L\ and\ f^U$ are continuous at c.

Definition 1. We define a linear fractional function F(x) as follows:

$$\text{F(x)} = \frac{cx+\alpha_1}{dx+\alpha_2} \quad (8)$$

Where x = $(x_1, x_2, \dots, x_n)^t \in \Re^n$, c = $(c_1, c_2, \dots, c_n) \in \Re^n$, d = $(d_1, d_2, \dots, d_n) \in \Re^n$, and α_1, α_2 are real scalars.

Definition 2. Let $F : V \to I$ be an interval-valued function, defined on a real vector space V. Now, we consider the following interval-valued optimization problem

$$\text{(IVP)} \quad \text{Min}_{\preccurlyeq}\ F(x) = [F_L(x), F_U(x)]$$
$$\text{Subject to x} \in \text{Y} \subseteq \text{V}, \quad (9)$$

where Y is the feasible set of problem (IVP) and ≼ is the partial ordering on the set of integers.

Definition 3. Let ≼ be a partial ordering on the set of integers. Then, for $a = [a^L,\ a^U]$, and $b = [b^L, b^U]$, we write

$$a \preccurlyeq b$$
$$\text{if and only if } a^L \leq b^L \text{ and } a^U \leq b^U.$$

This means that a is inferior to b or b is superior to a.

Definition 4. Let F :V →I be an interval-valued function, defined on a real vector space V, and let Y be a subset of V. Suppose that we are going to maximize F. We say that F $(\bar{x})$ is a non-dominated objective value if and only if there exists no x $(\neq \bar{x}) \in Y$ such that $F(\bar{x}) \prec F(x)$.

Interval-Valued Linear Fractional Programming (IVLFP)

$$\begin{aligned} &\text{Minimize} \quad f(x) = \frac{cx+\alpha_1}{dx+\alpha_2} \\ &\text{Subject to} \quad Ax = b \\ &\qquad x \geq 0 \end{aligned} \tag{10}$$

Suppose c = $(c_1, c_2, \dots, c_n)$, d = $(d_1, d_2, \dots, d_n)$, where $c_j, d_j \in I$, j = 1,2, ..., n.
Define $c^L = (c_1^L, c_2^L, \dots, c_n^L)$, $c^U = (c_1^U, c_2^U, \dots, c_n^U)$, $d^L = (d_1^L, d_2^L, \dots, d_n^L)$, $d^U = (d_1^U, d_2^U, \dots, d_n^U)$, where $c_j^L, c_j^U, d_j^L, d_j^L$ are real scalars for j = 1,2, ..., n. Also, $\alpha_1 = [\alpha_1^L,\ \alpha_1^U]$, $\alpha_2 = [\alpha_2^L,\ \alpha_2^U]$.

Then IVLFP (10) can be re-written as:

$$\begin{aligned} &\text{Minimize} \quad f(x) = \frac{p(x)}{q(x)} \\ &\text{Subject to} \quad Ax = b \\ &\qquad x \geq 0, \end{aligned} \tag{11}$$

where p(x) and q(x) are interval-valued linear functions and are given by

$$p(x) = [p^L(x), p^U(x)] = [c^L x + \alpha_1^L, c^U x + \alpha_1^U],$$
$$q(x) = [q^L(x), q^U(x)] = [d^L x + \alpha_2^L, d^U x + \alpha_2^U].$$

Therefore, IVLFP (11) can re-written as

$$\begin{aligned} &\text{Minimize} \quad f(x) = \frac{[c^L x+\alpha_1^L,\ c^U x+\alpha_1^U]}{[d^L x+\alpha_2^L,\ d^U x+\alpha_2^U]} \\ &\text{Subject to} \quad Ax = b \\ &\qquad x \geq 0 \end{aligned} \tag{12}$$

Further, using the concept of division of two interval-valued numbers, IVLFP (12) can re-written as

$$\begin{aligned} &\text{Minimize} \quad f(x) = [f^L(x), f^U(x)] \\ &\text{Subject to} \quad Ax = b \\ &\qquad x \geq 0 \end{aligned} \tag{13}$$

where $f^L(x)$ and $f^U(x)$ are linear fractional functions.

Definition 5. (Wu (2008)) Let x^* be a feasible solution of IVLFP (13). Then, x^* is a non-dominated solution of IVLFP (13) if there exist no feasible solution x such that $f(x) \prec f(x^*)$. In this case, we say that $f(x^*)$ is the non-dominated objective value of f.

Corresponding to IVLFP (13)), consider the following optimization problem:

$$\begin{aligned} &\text{Minimize} \quad g(x) = f^L(x) + f^U(x) \\ &\text{Subject to} \quad Ax = b \\ &\qquad x \geq 0 \end{aligned} \tag{14}$$

To solve IVLFP (14), we use the following theorem (Wu (2008)).

Theorem. Wu (2008) If x^* is an optimal solution of IVLFP(14), then x^* is a non-dominated solution of IVLFP(13).

Proof. Let us refer to Wu(2008).

3. NOTATIONS AND ASSUMPTIONS

To develop the proposed model, we considered the following notations and assumptions:

Notations

λ : Fixed cost per order
B : Maximum available budget for all products
W : Maximum available space for all products
For i^{th} product: (i = 1, 2, …, n)
Q_i : Ordering quantity of product *i,*
h_i : Holding cost per product per unit time for i^{th} product
P_i : Purchasing price of i^{th} product
S_i : Selling price of i^{th} product
D_i : Demand quantity per unit time of i^{th} product
f_i : Space required per unit for i^{th} product
OC_i : Ordering cost of i^{th} product.

Assumptions

1. Multiple products are considered in this model.
2. Infinite time horizon is considered. Further, there is only one period in the cycle time.
3. Lead time is zero, and so rate of replenishment is infinite.
4. Holding cost is known and constant for each product.
5. Demand is taken here as inversely related to the selling price of the product, that is,
$$D_i = D_i(S_i) = m_1 S_i^{-m_2},$$
where $m_1 > 0$ is a scaling constant, and $m_2 > 1$ is price-elasticity coefficient. For notational simplicity,
we will use D_i and $D_i(S_i)$ interchangeably.
6. Shortages are not allowed.
7. Purchase price is constant for each product, i.e. no discount is available.

4. CRISP LINEAR FRACTIONAL PROGRAMMING INVENTORY PROBLEM

A multi-product inventory model under resources constraints is introduced as a linear fractional programming problem. This model refers to a multi-product inventory problem, with limited capacity of warehouse and constraints on investment in inventories. For each product, we impose the constraint on ordering cost.

We practically observe the inventory models with more than one objective functions. These objectives may be in conflict with each other, or may not be. In such type of inventory models, the decision maker is interested to maximize or minimize two or more objectives simultaneously over a given set of decision variables. We call this model as linear fractional inventory model. Without loss of generality, we assume there is only one period in the cycle time. Then,

$$\text{Total Profit} = \sum_{i=1}^{n} (S_i - P_i) Q_i \tag{15a}$$

$$\text{Holding cost} = \sum_{i=1}^{n} \frac{h_i Q_i}{2} \tag{15b}$$

$$\text{Ordering cost} = \sum_{i=1}^{n} \frac{\lambda D_i}{Q_i} = \sum_{i=1}^{n} \frac{\lambda m_1 S_i^{-m_2}}{Q_i} \quad (15c)$$

$$\text{Back ordered quantity} = \sum_{i=1}^{n} (D_i - Q_i) = \sum_{i=1}^{n} (m_1 S_i^{-m_2} - Q_i) \quad (15d)$$

$$\text{Total ordered quantity} = \sum_{i=1}^{n} Q_i \quad (15e)$$

In this paper, the crisp linear fractional programming inventory model is formulated as

Maximize: $Z = \frac{\text{Total Profit}}{\text{Holding cost}}$,
Subject to, (i) Total Budget Limit,
(ii) Storage Space Limit,
(iii) Limit on Ordering Cost of Each Product.
(iv) Non-Negative Restriction.

Formulation of Constraints

(i) **Total Budget Limit:** A budget constraint is all about the combinations of goods and services that a consumer may purchase given current prices within his or her given income. For the proposed model, the budget constraint is

$$\sum_{i=1}^{n} P_i Q_i \leq B \quad (16a)$$

(ii) **Storage Space Limit:** $\sum_{i=1}^{n} f_i Q_i \leq W$ (16b)

(iii) **Limit on Ordering Cost of Each Product:** We impose the upper limit of ordering cost as a constraint. Since OC_1, OC_2, …, OC_n are the ordering cost of 1st product, 2nd product,…, nth product, we can express the concerned constrained as follows:

For first product, $\frac{\lambda D_1}{Q_1} \leq OC_1$

$\Rightarrow \lambda D_1 - OC_1 . Q_1 \leq 0$

For second product, $\frac{\lambda D_2}{Q_2} \leq OC_2$

$\Rightarrow \lambda D_2 - OC_2 . Q_2 \leq 0$

Similarly, for nth product, $\lambda D_n - OC_n . Q_n \leq 0$

$$\Rightarrow \lambda m_1 S_n^{-m_2} - OC_n . Q_n \leq 0 \quad (16c)$$

(iv) **Non-Negative Restriction:**

$$Q_1, Q_2, \ldots, Q_i \geq 0 \quad (16d)$$

5. INTERVAL-VALUED LINEAR FRACTIONAL PROGRAMMING INVENTORY PROBLEM

In real life cases, it is observed that the selling price is uncertain due to dynamic behavior of the market. So it may be considered to vary in an interval. Let the selling price be represented by the interval $[S_{iL}, S_{iU}]$, where S_{iL} & S_{iU} are the left limit & right limit of interval for S_i. Mathematically, we can express as

$$S_i = [S_{iL}, S_{iU}], \text{ with } S_{iL} \leq S_{iU} \quad (17)$$

Similarly, the purchasing, holding cost, and demand would be expressed as interval-valued numbers as

$$P_i = [P_{iL}, P_{iU}], \text{ with } P_{iL} \leq P_{iU} \tag{18}$$

$$h_i = [h_{iL}, h_{iU}], \text{ with } h_{iL} \leq h_{iU} \tag{19}$$

$$D_i = [m_1 S_{iL}^{-m_2}, m_1 S_{iU}^{-m_2}], \text{ with } m_1 S_{iL}^{-m_2} \leq m_1 S_{iU}^{-m_2} \tag{20}$$

Accordingly, the constraints transform to the interval-valued form as follows

Total Budget Limit: $\sum_{i=1}^{n} [P_{iL}, P_{iU}] Q_i \leq B$ (21)

Storage Space Limit: $\sum_{i=1}^{n} f_i Q_i \leq W$ (22)

Limit on Ordering Cost of each Product*:*

For n^th product, we have

$$\lambda [m_1 S_{nL}^{-m_2}, m_1 S_{nU}^{-m_2}] \leq Q_n [OC_{nL}, OC_{nU}] \tag{23}$$

Non-Negative Restriction:

$$Q_1, Q_2, \ldots, Q_i \geq 0 \tag{24}$$

In this case, we define the following interval-valued fuzzy linear fractional programming problem for the proposed inventory model,

(IVP1) Maximize: $Z = \dfrac{\sum_{i=1}^{n} ([S_{iL}, S_{iU}] - [P_{iL}, P_{iU}]) Q_i}{\sum_{i=1}^{n} \frac{[h_{iL}, h_{iU}] Q_i}{2}}$, (25)

Subject to $\sum_{i=1}^{n} [P_{iL}, P_{iU}] Q_i \leq B$,

$\sum_{i=1}^{n} f_i Q_i \leq W$,

$Q_i [OC_{iL}, OC_{iU}] \geq \lambda [m_1 S_{nL}^{-m_2}, m_1 S_{nU}^{-m_2}]$, (for i = 1, 2,.., n)

and $Q_i \geq 0$, (for i = 1, 2,.., n)

Using the arithmetic operations of intervals, the interval-valued fractional objective function can be written as

$$\begin{aligned} Z &= \frac{\sum_{i=1}^{n} [S_{iL} - P_{iU},\ S_{iU} - P_{iL}] Q_i}{\sum_{i=1}^{n} [\frac{h_{iL} Q_i}{2},\ \frac{h_{iU} Q_i}{2}]} \\ &= \frac{[\sum_{i=1}^{n} (S_{iL} - P_{iU}) Q_i,\ \sum_{i=1}^{n} (S_{iU} - P_{iL}) Q_i]}{[\sum_{i=1}^{n} \frac{h_{iL} Q_i}{2},\ \sum_{i=1}^{n} \frac{h_{iU} Q_i}{2}]} \\ &= [\frac{\sum_{i=1}^{n} (S_{iL} - P_{iU}) Q_i}{\sum_{i=1}^{n} \frac{h_{iU} Q_i}{2}}, \frac{\sum_{i=1}^{n} (S_{iU} - P_{iL}) Q_i}{\sum_{i=1}^{n} \frac{h_{iL} Q_i}{2}}] \\ &= [Z_L(Q_1,\ Q_2, \ldots, Q_n),\ Z_U(Q_1,\ Q_2, \ldots, Q_n)] \end{aligned} \tag{26}$$

where $Z_L(Q_1,\ Q_2, \ldots, Q_n) = \dfrac{\sum_{i=1}^{n} (S_{iL} - P_{iU}) Q_i}{\sum_{i=1}^{n} \frac{h_{iU} Q_i}{2}}$,

and $Z_U(Q_1,\ Q_2, \ldots, Q_n) = \dfrac{\sum_{i=1}^{n} (S_{iU} - P_{iL}) Q_i}{\sum_{i=1}^{n} \frac{h_{iL} Q_i}{2}}$ (27)

Thus the single-objective function consisting of interval-valued number coefficients is transformed into multi-objective functions given by (27) where, in each objective function, the coefficients are crisp numbers. Hence, the interval-valued optimization problem (IVP1), transforms to following optimization problem as follows, (IVP2) Maximize$[Z_L(Q_1, Q_2, \ldots, Q_n)$, $Z_U(Q_1, Q_2, \ldots, Q_n)$ (28)

Subject to $\sum_{i=1}^{n}[P_{iL}, P_{iU}]Q_i \leq B$,
$\sum_{i=1}^{n} f_i Q_i \leq W$,
$Q_i[OC_{iL}, OC_{iU}] \geq \lambda[m_1 S_{iL}^{-m_2}, m_1 S_{iU}^{-m_2}]$, (for i = 1, 2,.., n),
And $Q_i \geq 0$, (for i = 1, 2,.., n).

To solution this problem, the techniques of classical linear fractional programming cannot be applied if and unless the above interval-valued structure of the problem be reduced into a standard linear fractional programming problem.

To deal with interval inequality constraints, we use the Tong's Approach [Tong (1994) & Sengupta et al (2001)]. Tong deals with interval inequality constraints in a separate way.

According to Tong's Approach, each interval inequality constraint is transformed into 2^{n+1} crisp inequalities. Let us take a simple inequality constraint with a single variable:

$$[10, 20]x \leq [5, 35].$$

According to Tong (1994), the interval inequality generates 2^{1+1} crisp inequalities:

$$10x \leq 5 \Rightarrow x \leq 0.5$$
$$10x \leq 35 \Rightarrow x \leq 3.5$$
$$20x \leq 5 \Rightarrow x \leq 0.25$$
$$20x \leq 35 \Rightarrow x \leq 1.75$$

Hence, the interval-valued linear fractional inventory problem (IVP2) can be transformed to a non-interval optimization problem as follows:

(IVP3) Maximize $[Z_L(Q_1, \ldots, Q_n), Z_U(Q_1, \ldots, Q_n)]$ (29)

where $Z_L(Q_1, Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n}(S_{iL} - P_{iU})Q_i}{\sum_{i=1}^{n}\frac{h_{iU}Q_i}{2}}$

$Z_U(Q_1, Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n}(S_{iU} - P_{iL})Q_i}{\sum_{i=1}^{n}\frac{h_{iL}Q_i}{2}}$

Subject to: $\sum_{i=1}^{n} P_{iL} Q_i \leq B$
$\sum_{i=1}^{n} P_{iU} Q_i \leq B$
$\sum_{i=1}^{n} f_i Q_i \leq W$
$OC_{iL} Q_i \geq \lambda m_1 S_{iL}^{-m_2}$, (for i = 1, 2,.., n)
$OC_{iU} Q_i \geq \lambda m_1 S_{iL}^{-m_2}$, (for i = 1, 2,.., n)
$OC_{iL} Q_i \geq \lambda m_1 S_{iU}^{-m_2}$, (for i = 1, 2,.., n)
$OC_{iU} Q_i \geq \lambda m_1 S_{iU}^{-m_2}$, (for i = 1, 2,.., n)
$Q_i \geq 0$, (for i = 1, 2,.., n)

Now, let us write the corresponding optimization problem of (IVP3) as follows:

(IVP4) Maximize g(x) $= Z_L(Q_1, Q_2, \ldots, Q_n) + Z_U(Q_1, Q_2, \ldots, Q_n)$ (30)

$$\text{Subject to } \sum_{i=1}^{n} P_{iL} Q_i \leq B$$
$$\sum_{i=1}^{n} P_{iU} Q_i \leq B$$
$$\sum_{i=1}^{n} f_i Q_i \leq W$$
$$OC_{iL} Q_i \geq \lambda m_1 S_{iL}^{-m_2}, \text{ (for i = 1, 2,.., n)}$$
$$OC_{iU} Q_i \geq \lambda m_1 S_{iL}^{-m_2}, \text{ (for i = 1, 2,.., n)}$$
$$OC_{iL} Q_i \geq \lambda m_1 S_{iU}^{-m_2}, \text{ (for i = 1, 2,.., n)}$$
$$OC_{iU} Q_i \geq \lambda m_1 S_{iU}^{-m_2}, \text{ (for i = 1, 2,.., n)}$$
$$Q_i \geq 0, \text{ (for i = 1, 2,.., n)}$$

To solve the (IVP3), theorem (Wu (2008)) is conveniently used. According to this theorem, the non-dominated solution of (IVP3), would be determined by solving its corresponding optimization problem (IVP4).

6. NUMERICAL EXAMPLES

To illustrate the proposed approach, we solve the following two numerical examples in crisp and interval-value cases:

Example1. **Crisp Case** Consider an inventory model with following input data (in proper units):

Table 1: Input data (Crisp Case)

Product	h_i	P_i	S_i	OC_i	f_i	m_1	m_2	λ	W	B
i = 1	12	125	200	80	2	80,000	1.20	7	300	90,000
i = 2	16	160	220	90	4					

$$\text{Maximize: } Z = \frac{\sum_{i=1}^{2}(S_i - P_i)Q_i}{\sum_{i=1}^{2}\frac{h_i Q_i}{2}} = \frac{75Q_1 + 60Q_2}{6Q_1 + 8Q_2}$$

$$\text{Subject to: } \sum_{i=1}^{n} P_i Q_i \leq B \quad \Rightarrow \quad 125Q_1 + 160Q_2 \leq 90000,$$
$$\sum_{i=1}^{n} f_i Q_i \leq W \quad \Rightarrow \quad 2Q_1 + 4Q_2 \leq 300,$$
$$OC_n . Q_n \geq \lambda m_1 S_n^{-m_2} \quad \Rightarrow \quad 80Q_1 \geq 970.4027, \text{ and } 90Q_2 \geq 865.5273.$$

The optimal solution is $Q_1^* = 130.7658$, $Q_2^* = 9.6170$, $Z^* = 12.0535$,
Maximum profit = 10384.455.

Interval-Values Case Consider an inventory model with following input data:

Table 2: Input data (Interval-Valued Case)

Product	h_i	P_i	S_i	OC_i	f_i	m_1	m_2	λ	W	B
i = 1	[10, 14]	[120, 130]	[190, 210]	[70, 90]	2	80,000	1.20	7	300	90,000
i = 2	[14, 18]	[150, 170]	[210, 230]	[80, 100]	4					

Objective: Maximize $Z(Q_i) = \frac{\sum_{i=1}^{n}([s_{iL},\ s_{iU}] - [p_{iL},\ p_{iU}])Q_i}{\sum_{i=1}^{n}\frac{[h_{iL},\ h_{iU}]Q_i}{2}}$

$$= \frac{([190,\ 210] - [120,\ 130])Q_1 + ([210,\ 230] - [150,\ 170])Q_2}{\frac{[10,\ 14]Q_1}{2} + \frac{[14,\ 18]Q_2}{2}},$$

$$Z_L(Q_1,\ Q_2, ..., Q_n) = \frac{\sum_{i=1}^{n}(s_{iL} - p_{iU})Q_i}{\sum_{i=1}^{n}\frac{h_{iU}Q_i}{2}} = \frac{(190-130)Q_1 + (210-170)Q_2}{7Q_1 + 9Q_2} = \frac{60Q_1 + 40Q_2}{7Q_1 + 9Q_2},$$

$$Z_U(Q_1,\ Q_2, ..., Q_n) = \frac{\sum_{i=1}^{n}(s_{iU} - p_{iL})Q_i}{\sum_{i=1}^{n}\frac{h_{iL}Q_i}{2}} = \frac{(210-120)Q_1 + (230-150)Q_2}{5Q_1 + 7Q_2} = \frac{90Q_1 + 80Q_2}{5Q_1 + 7Q_2}.$$

The corresponding optimization problem would be:

Maximize $g(x) = \frac{60Q_1 + 40Q_2}{7Q_1 + 9Q_2} + \frac{90Q_1 + 80Q_2}{5Q_1 + 7Q_2}$,

Subject to: $\sum_{i=1}^{n} P_{iL}Q_i \leq B \quad \Rightarrow 120Q_1 + 150Q_2 \leq 90000,$

$\sum_{i=1}^{n} P_{iU}Q_i \leq B \quad \Rightarrow 130Q_1 + 170Q_2 \leq 90000,$

$\sum_{i=1}^{n} f_iQ_i \leq W \quad \Rightarrow 2Q_1 + 4Q_2 \leq 300,$

$OC_{iL}Q_i \geq \lambda m_1 S_{iL}^{-m_2} \quad \Rightarrow 70Q_1 \geq 1032.0095,\ \ 80Q_2 \geq 915.2187,$

$OC_{iU}Q_i \geq \lambda m_1 S_{iL}^{-m_2} \quad \Rightarrow 90Q_1 \geq 1032.0095,\ \ 100Q_2 \geq 915.2187,$

$OC_{iL}Q_i \geq \lambda m_1 S_{iU}^{-m_2} \quad \Rightarrow 70Q_1 \geq 915.2187,\ \ 80Q_2 \geq 820.5681,$

$OC_{iU}Q_n \geq \lambda m_1 S_{iU}^{-m_2} \quad \Rightarrow 90Q_1 \geq 915.2187,\ \ 100Q_2 \geq 820.5681$

and $Q_1 \geq 0$, $Q_2 \geq 0$

The optimal solution is $Q_1^* = 127.1197,\ \ Q_2^* = 11.4402$ with optimum value $g(Q_1^*, Q_2^*) = 25.4081$.

Maximum Profit $= ([190, 210] - [120, 130])Q_1^* + ([210, 230] - [150, 170])Q_2^*$

$= [60,\ 90]Q_1^* + ([40,\ 80]Q_2^*$

$= [60Q_1^* + 40Q_2^*,\ 90Q_1^* + 80Q_2^*]$

$= [8084.79, 12355.989]$.

Hence, the bounds of profit are 8084.79 (lower bound), and 12355.989 (upper bound).

Example2. **<u>Crisp Case</u>** Consider an inventory model with following input data (in proper units):

Table 3: Input data (Crisp Case)

Product	h_i	P_i	S_i	OC_i	f_i	m_1	m_2	λ	W	B
i = 1	12	125	180	70	1	15,000	1.10	5	400	60,000
i = 2	15	150	220	80	2					

Maximize: $Z = \frac{\sum_{i=1}^{n}(S_i - P_i)Q_i}{\sum_{i=1}^{n}\frac{h_iQ_i}{2}} = \frac{55Q_1 + 70Q_2}{6Q_1 + 7.5Q_2}$

Subject to: $\sum_{i=1}^{n} P_iQ_i \leq B \quad \Rightarrow 125Q_1 + 150Q_2 \leq 60000,$

$\sum_{i=1}^{n} f_iQ_i \leq W \quad \Rightarrow Q_1 + 2Q_2 \leq 400,$

$OC_n.Q_n \geq \lambda m_1 S_n^{-m_2} \Rightarrow 70Q_1 \geq 247.8914$, and $80Q_2 \geq 198.7908$.

The optimal solution is $Q_1^* = 3.5413,\ \ Q_2^* = 198.2293,\ Z^* = 9.3310$,

Maximum profit = 14070.8225.

Interval-Values Case Consider an inventory model with following input data:

Table 4: Input data (Interval-Valued Case)

Product	h_i	P_i	S_i	OC_i	f_i	m_1	m_2	λ	W	B
i = 1	[8, 16]	[100, 150]	[160, 200]	[60, 80]	1	15,000	1.10	5	400	60,000
i = 2	[12, 18]	[125, 175]	[200, 240]	[70, 90]	2					

Objective: Maximize $Z(Q_i) = \frac{\sum_{i=1}^{n}([S_{iL}, \ S_{iU}] - [P_{iL}, \ P_{iU}])Q_i}{\sum_{i=1}^{n}\frac{[h_{iL}, \ h_{iU}]Q_i}{2}}$

$$= \frac{([160, \ 200] - [100, \ 150])Q_1 + ([200, \ 240] - [125, \ 175])Q_2}{\frac{[8, \ 16]\,Q_1}{2} + \frac{[12, \ 18]\,Q_2}{2}},$$

$$Z_L(Q_1, \ Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n}(S_{iL} - P_{iU})Q_i}{\sum_{i=1}^{n}\frac{h_{iU}Q_i}{2}} = \frac{(160-150)Q_1 + (200-175)Q_2}{8Q_1 + 9Q_2} = \frac{10Q_1 + 25Q_2}{8Q_1 + 9Q_2}$$

$$Z_U(Q_1, \ Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n}(S_{iU} - P_{iL})Q_i}{\sum_{i=1}^{n}\frac{h_{iL}Q_i}{2}} = \frac{(200-100)Q_1 + (240-125)Q_2}{4Q_1 + 6Q_2} = \frac{100Q_1 + 115Q_2}{4Q_1 + 6Q_2}.$$

The corresponding optimization problem would be:

$$\text{Maximize } g(\text{x}) = \frac{10Q_1 + 25Q_2}{8Q_1 + 9Q_2} + \frac{100Q_1 + 115Q_2}{4Q_1 + 6Q_2},$$

Subject to:
$\sum_{i=1}^{n} P_{iL}Q_i \le B \quad \Rightarrow 100Q_1 + 125Q_2 \le 60000,$
$\sum_{i=1}^{n} P_{iU}Q_i \le B \quad \Rightarrow 150Q_1 + 175Q_2 \le 60000,$
$\sum_{i=1}^{n} f_i Q_i \le W \quad \Rightarrow Q_1 + 2Q_2 \le 400,$
$OC_{nL}Q_n \ge \lambda m_1 S_{iL}^{-m_2} \quad \Rightarrow 60Q_1 \ge 282.1819, \ 70Q_2 \ge 220.7640,$
$OC_{nU}Q_n \ge \lambda m_1 S_{iL}^{-m_2} \quad \Rightarrow 80Q_1 \ge 282.1819, \ 90Q_2 \ge 220.7640,$
$OC_{nL}Q_n \ge \lambda m_1 S_{iU}^{-m_2} \quad \Rightarrow 60Q_1 \ge 220.7640, \ 70Q_2 \ge 180.6462,$
$OC_{nU}Q_n \ge \lambda m_1 S_{iU}^{-m_2} \quad \Rightarrow 80Q_1 \ge 220.7640, \ 90Q_2 \ge 180.6462,$
and $Q_1 \ge 0, \ Q_2 \ge 0.$

The optimal solution is $Q_1^* = 393.6925, \ Q_2^* = 3.1538$ with optimum value $g(Q_1^*, Q_2^*) = 26.1944.$
Maximum Profit $= (([160, 200] - [100, 150])Q_1^* + ([200, 240] - [125, 175])Q_2^*)$
$= [10, 100]Q_1^* + ([75, 115]Q_2^*$
$= [10Q_1^* + 75Q_2^*, \ 100Q_1^* + 115Q_2^*]$
$= [4173.46, 39731.937].$

Hence, the bounds of profit are 4173.46 (lower bound), and 39731.937 (upper bound).

7. CONCLUSIONS

In this paper, the author presented an approach to solve interval-valued inventory optimization problem based on linear fractional programming. The uncertainty in inventory parameters is represented by interval-valued numbers. Using interval arithmetic, the interval-valued optimization problem is changed into a crisp multi-objective linear fractional programming problem. LINGO package is used to solve the subsequent optimization problem. The optimal order quantity and profit bounds are determined.

For future directions, the paper can be extended to inventory model with shortages case and inventory model with price-discount. Moreover, it can also be extended to interval-valued multi-objective optimization case.

REFERENCES

[1] Alefeld, G., Herzberger, J., (1983), Introduction to interval computations. Academic Press, London.
[2] Bhurjee, A.K., Panda, G., (2012), Efficient solution of interval optimization problem. Mathematical Methods of Operation Research, 76(3), 273–288.
[3] Carla Oliveira, Carlos Henggeler Antunes, (2007), Multiple objective linear programming models with interval coefficients – an illustrated overview. European Journal of Operational Research, 181(3), 1434–1463.
[4] Chanas, S., Kuchta, D., (1996), Multi-objective programming in optimization of interval objective functions-a generalized approach. European Journal of Operational Research, 94(3), 594–598.
[5] Charnes, A., Cooper, W.W., (1962), Programming with linear fractional functions. Naval Research Logistics. Quarterly, 9(3-4), 181–186.
[6] Charnes, A., Granot, F., Phillips, F., (1977), An algorithm for solving interval linear programming problems. Operations Research, 25(4), 688–695.
[7] Chinneck, J.W., Ramadan, K., (2000), Linear programming with interval coefficients. Journal of Operational Research Society, 51(2), 209–220
[8] Deepmala, (2014), A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India – 492010.
[9] Effati, S., Pakdaman, M., (2012), Solving the interval-valued linear fractional programming problem. American Journal of Computational Mathematics, 2(1), 51–55.
[10] Ishibuchi, H.,Tanaka, H., (1990), Multi objective programming in optimization of the interval objective function. European Journal of Operational Research, 48(2), 219–225.
[11] Kumar, P., Dutta, D., (2015), Multi-objective Linear Fractional Inventory Model of Multi-Products with Price-Dependant Demand Rate in Fuzzy Environment, International Journal of Mathematics in Operations Research : Inderscience Journal, 7(5), 547-565.
[12] Levin, V.I., (1999), Nonlinear optimization under interval uncertainty. Cybernetics and Systems Analysis, 35(2), 297–306.
[13] Liu, S.T., (2006), Computational method for the profit bounds of inventory model with interval demand and unit cost. Applied Mathematics and Computation, 183(1), 499–507.
[14] Liu, S.T., Wang, R.T., (2007), A numerical solution method to interval quadratic programming. Applied Mathematics and Computation, 189(2), 1274–1281.
[15] Liu, S.T., (2008), Posynomial geometric programming with interval exponents and coefficients. European Journal of Operational Research, 186(1), 17–27.
[16] Mishra, V. N., (2007), Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis, Indian Institute of Technology, Roorkee - 247 667, Uttarakhand, India.
[17] Moore, R., Lodwick, W., (2003), Interval analysis and fuzzy set theory. Fuzzy Sets and Systems, 135(1), 5–9.
[18] Mráz, F., (1998), Calculating the exact bounds of optimal values in LP with interval coefficients. Annals of Operation Research, 81(1), 51–62.
[19] Sadjadi, S.J., Aryanezhad, M.B., Sarfaraj, A., (2005), A fuzzy approach to solve a multi-objective linear fractional inventory model. Journal of Industrial Engineering International, 1(1), 43–47.
[20] Sengupta, A., Pal, T.K., Chakraborty, D., (2001), Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming. Fuzzy Sets and Systems, 119(1), 129–138.
[21] Steuer, R.E., (1981), Algorithms for linear programming problems with interval objective function coefficients. Mathematics of Operation Research, 6(3), 333–348.
[22] Toksarı, M.D., (2008), Taylor series approach to fuzzy multi objective linear fractional programming. Information Sciences, 178(4), 1189–1204.
[23] Tong, S., (1994), Interval number and fuzzy number linear programming. Fuzzy Sets and Systems, 66(3), 301–306.
[24] Vandana., Sharma, B.K., (2015), An EPQ inventory model for non-instantaneous deteriorating item under trade credit policy, International Journal of Mathematical Science & Engineering Applications, (IJMSEA), 9(1), 179-188.

[25] Wu, H.C., (2007), The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function. European Journal of Operational Research, 176(1), 46–59.
[26] Wu, H.C., (2008a), On interval-valued nonlinear programming problems. Journal of Mathematical Analysis and Applications, 338(1), 299–316.
[27] Wu, H.C., (2008b), Wolfe duality for interval-valued optimization. Journal of Optimization Theory and Applications, 138(3), 497–509.
[28] Wu, H.C., (2010), Duality theory for optimization problems with interval-valued objective functions. Journal of Optimization Theory and Applications, 144(3), 615–628.