Solving Linear Fractional Programming Problems Using a New Homotopy Perturbation Method

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Abstract

A new Homotopy Perturbation Method (HPM) is used to find exact solutions for the system of Linear Fractional Programming Problem (LFPP) with equality constraints. In best of my knowledge, first time we are going to introduce a new technique using Homotopy for solving LFP problem. The Homotopy Perturbation method (HPM) and factorization technique are used together to build a new method. A new technique is also used to convert LFPP to Linear programming problem (LPP). The results betray that our proposed method is very easy and effective compare to the existing method for solving LFP problems with equality constraints applied in real life situations. To illustrate the proposed method numerical examples are solved and the obtained results are discussed.

Keywords

Homotopy Perturbation Method, Factorisation, LFPP, LPP

1. Introduction

The Linear Fractional Programming Problems (i.e. objective function has a numerator and a denominator) have attracted many researchers due to its real life application in many important field such as health care, financial sector, production planning and hospital planning. Charnes-Cooper [3], have proposed a method which depends on transforming the LFPP to an equivalent linear program. Bitran and Magnant [2] have concerned duality and sensitivity analysis in LFP .An iterative method for solving problem is based on conjugate gradient projection method [17]. Swarup [16] developed a simplex technique for the same problem.

In 1998 non-linear partial differential equation, non-linear systems of second order boundary value problems to solve integral, functional integral and also system of linear equations [5] are solved by Homotopy Perturbation Method (HPM) was proposed by He. An efficient algorithm for solving of linear equations based on HPM [7].The Homotopy Perturbation Method (HPM) by He in 1998, is very systematic in application and has been universally used for solving linear equations. In 1992, Liao gave a crowd ideas of Homotopy analysis for the solution of nonlinear problems. The basic ideas of HPM has been successfully applied to solve many types of non-linear problems. Using HPM, gives a very rapid convergence of the solution series. Many researcher’s used this method to solve linear equations.FirstlyH.Saberi Najafi [15] introduced

HPM for solving linear programming problems. In 2011, M. Mehrabinezhad [14] gave application of He's HPM to linear programming problems.

First, we applied a new technique to convert LFPP to LPP and introduced HPM to solve LPP. A new method is obtained (with the help of HPM and factorisation method) to solve the LP problem. We use HPM and factorization method to get a new method and solve Linear Programming (LP) problem. We devote section 2 for the basic idea of our technique to convert LFPP to LPP. In section 3 a brief review of HPM for solving LPP has been discussed. Section 4 shows the effectiveness of our method discusses via certain examples. Lastly, in section 5, we conclude our results and compare them with the original results.

2. Analysis of LFPP

The Linear Fractional Programming (LFP) problem is stated as:

$$\text{Maximize } F(x) = \frac{c^t x + \alpha}{d^t x + \beta} \tag{2.1}$$

Subject to constraints Ax=b,
$0 \geq x$

Where x, c and d are nx1 vectors, A is an $n\ mx$ matrix, A= $({}_{ij}a\ )$ (I =1,……..,m: j=1,……..n), b is an mx1 vectors and β α, are scalars. We point out that the denominators are non-negative. Now we separate LFP problem to LP problem.

$$\text{Maximize } F(x) = c^t x + \alpha \tag{2.2}$$

$$\text{Subject to } d^t x + \beta = \theta$$

$$Ax = b$$
$$x \geq 0$$

$$\text{Where } \theta \text{ is defined by Minimize } F(x) = d^t x + \beta \tag{2.3}$$

Subject to Ax=b,
x>=0

3. HPM Strategies

Consider a linear equations in the following form:

$$Ax=b \tag{3.1}$$

Where A= [*ij a*], b= [*ij b*], x= [*x j*], i =1, 2…………n, j=1, 2…………m

Let L (u) =Au-b=0 with solution u=x and F (u) =u (3.2)

Then we consider a convex Homotopy H (u, p) as follows:

$$H (u, p) = (1-p) F (u) +p L (u) =0 \tag{3.3}$$

The convex Homotopy continuously trace an implicity defined curve from a starting point H(u,o) to a solution function H(u,1).The embedding parameter p monotonically increases from zero to unit as trival problem F(u)=0 is continuously deformed to original problem L(u)=0.The embedding parameter p□ [0,1] can be considered as an expanding parameter in equations [3.1-3.3]

Ultimately, we have H (u, 0) =F (u), H (u, 1)=L(u)

The solution of equations can be written as a power series in parameter p.

$$U=u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{3.4}$$

When p -> 1, then the approximate solution of equation (3.4) we get,

$$X=\lim\,(u_0 + pu_1 + p^2u_2 + \dots\dots\dots\dots\dots) = \sum_{k=0}^{\infty} u_k$$

Putting equations (3.1), (3.2), we substitute (3.4) in equation (3.3) and comparing the coefficient of identical powers of p, we find,

$$p^0 = u_0 = 0$$
$$p^1 = (A-I)u_0 + u_1 - b = 0, u_1 = b - (A-I)u_0$$
$$p^2 = (A-I)u_1 + u_2 = 0, u_2 = -(A-I)u_1$$

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The general term written as,

$$u_{n+1} = -(A-I)u_n, n = 1,2..............$$

Let, $u_0 = 0$, then we get,

$$u_1 = b$$
$$u_2 = -(A-I)b$$
$$u_3 = (A-I)^2 b$$

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$$u_{n+1} = (-1)^n (A-I)^n b$$

Then, the solution can be of the form,

$$u = u_0 + u_1 + u_2 + \dots\dots\dots\dots\dots\dots\dots$$

Or

$$u = [I - (A - I) + (A - I)^2 - \dots]\, b \tag{3.5}$$

The convergence of the series (3.5) has been proved by Keramati in [12]

3.1. Analysis of the new method of HPM

To describe the method, consider the linear equation in the following in general form,

$$AX=b \tag{3.1.1}$$

Where A is a matrix, x is variable, b is known functions (constraints). A general speaking can be divided in to two parts L and U

A=LU

Where L=lower triangular matrix, U= upper triangular matrix.

Equation (3.1.1) can be written in the form

LUX=b

Let UX=V, then LV=b (3.1.2)

Using, Homotopy technique to find the value of V.

Now ,we introduced a scheme to accelerate the rate of HPM allowed system of linear equations and defined a new Homotopy H (u, p, m, m) by,

H (u, 0, m, m) = F (u)

H (u, 1, m, m) = L (u) (3.1.3)

Now consider a convex Homotopy equations

H(u, p, m, m) = (1-p)F(u)+PL(u)+p(1-p)m+ p(1-p)m=0 (3.1.4)

Where m is called the accelerating vector if m=0 then H (u, p, 0) =H (u, p), which is the standard HPM.

By substituting (3.4) in equation (3.1.3) and equating the coefficients of identical powers of p, We have,

$$p^0 : v_0 = 0$$
$$p^1 : v_1 - v_0 + Lv_0 - b + 2m = 0, v_1 = b - (L - I)v_0 - 2m$$
$$p^2 : v_2 - v_1 + Lv_1 - 2m = 0, v_2 = -(L - I)v_1 + 2m$$
$$p^3 : v_3 = -(L - I)v_2$$

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$$p^{n+1} : v_{n+1} = -(L - I)v_n$$

Let $u_0 = 0$ then,

$$\begin{aligned} v_1 &= b - 2m \\ v_2 &= 2Lm - (L - I)b \\ v_3 &= -(L - I)v_2 \\ &. \\ &. \\ &. \\ v_{n+1} &= -(L - I)u_n \end{aligned} \tag{3.1.5}$$

Now find the parameters m such that v_2=0

$$2Lm-(L-I)b=0$$
$$2Lm=(L-I)b \qquad (3.1.6)$$

Thus from (3.1.4), we have $V=v_0+v_1+v_2+..........$

$$V=v_1 \text{ As } v_2, v_3.....=0 \qquad (3.1.7)$$

In the following we are going to introduce a new algorithm to find the exact solution of LFP problem with equality constraints. The steps of the proposed algorithm are given as follows:

Step 1. Formulate the problem of LFP as follows:

$$\text{Maximize } F(x)=\frac{c^t x+\alpha}{d^t x+\beta}$$

$$\text{Subject to} \quad Ax=b,$$

$$x \geq 0$$

Step 2. We separate the above LFP problem into two linear programming (LP) problem as:

$$\text{Maximize } F(x)=c^t x+\alpha$$

$$\text{Subject to } d^t x+\beta=\theta$$

$$Ax=b$$
$$x \geq 0$$

Where θ is defined by

$$\text{Minimize } F(x) = d^t x + \beta$$

$$\text{Subject to} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

Step 3. Using HPM and factorisation technique we solved L and U.

Step 4. Using step 3 and with respect to definitions 3.1.7 we get V.

Step 5. Based on equation 3.1.2 we solved UX=V.

Step 6. Using step 5 we find the solution of x1, x2, x3.

Step 7. Putting these values in step 1 we get the solution of objective value.

4. Numerical application

In this section we experiments two numerical example one is real life problem and other is numerical example.

Example 4.1. (Production Planning) [4]

A company manufactures two kinds of products A and B with profit around 5 and around 3 dollar per unit, respectively. However the cost for each one unit of the above products is around 5 and around 2 dollars respectively. It is assume that a fixed cost of around 1 dollar is added to the cost function due to expected duration through the process of production. Suppose the raw material needed for manufacturing product A and B is about 3 units per pound and about 5 units per pound respectively, the supply for this raw material is restricted to about 15 pounds. Man-hours per unit for the product A is about 5 hour and product B is about 2 hour per unit for manufacturing but total Man-hour available is about 10 hour daily. Determine how many products A and B should be manufactured in order to maximize the total profit.

We formulate this real life production planning problem in to LFP problem as follows:

$$\text{Maximize } F(x) = \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1} \quad (4.1.1)$$

$$\text{Subject to } \begin{array}{l} 3x_1 + 5x_2 = 15 \\ 5x_1 + 2x_2 = 10 \\ x_1, x_2 \geq 0 \end{array}$$

To separate the LFP problem and written as

$$\text{Maximize } F(x) = 5x_1 + 3x_2$$

$$\text{Subject to } \begin{array}{l} 5x_1 + 2x_2 + 1 = \theta \\ 3x_1 + 5x_2 = 15 \\ 5x_1 + 2x_2 = 10 \end{array}$$

(4.1.2)

Where θ is defined by Minimize $F(x) = 5x_1 + 2x_2 + 1$

$$\text{Subject to } \begin{array}{l} 3x_1 + 5x_2 = 15 \\ 5x_1 + 2x_2 = 10 \\ x_1, x_2 \geq 0 \end{array}$$

We define θ value

By using HPM and Factorisation method we have,

$$L=\begin{bmatrix}1 & 0\\ 5/3 & 1\end{bmatrix} \quad U=\begin{bmatrix}3 & 5\\ 0 & -19/3\end{bmatrix},\ b=\begin{bmatrix}15\\ 10\end{bmatrix}$$

Using equation (3.1.6) we obtain,

$$m_1=0, m_2=25/2$$

Thus the solution of equation (3.1.7) becomes

Thus the solution of equation (3.1.7) becomes V= v_1 =b-2m= $[15,10]^t-[0,25]^t=[15,-15]^t$

Thus from equation (3.1.2) UX=V and solution becomes $x_1=0.53, x_2=2.68$ Min z=9.01

Hence $\theta=9.01$ put the value in equation (2.6) we have,

$$L=\begin{bmatrix}1 & 0 & 0\\ 3/5 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},\ U=\begin{bmatrix}5 & 2 & 0\\ 0 & 19/5 & 0\\ 0 & 0 & 0\end{bmatrix},\ b=\begin{bmatrix}8.01\\ 15\\ 10\end{bmatrix}$$

Using equation (3.1.6) we have,

$$m_1=0, m_2=2.403, m_3=4.005$$

Thus the solution of (3.1.7) becomes $[8.01,15,10,]^t-[0,4.806,8.01]^t=[8.01,10.194,1.99]^t$

Thus from equation (3.1.2) the value of $x_1=0.53, x_2=2.68, x_3=0$

We settle the above value in equation (4.1.1) we earn the result of objective value is 1.18, Which is the exact solution. By comparing the result of proposed method with Swarup method, we conclude that our method is more reliable, because:

$$(Z)_{\text{proposed method}} = 1.18$$

$$(Z)_{\text{Swarup method}} = 1.28$$

$$1.18 = (Z)_{\text{proposed method}} < (Z)_{\text{Swarup method}} = 1.28$$

To show the compare result of proposed method with Swarup method we also present a graphical representation:

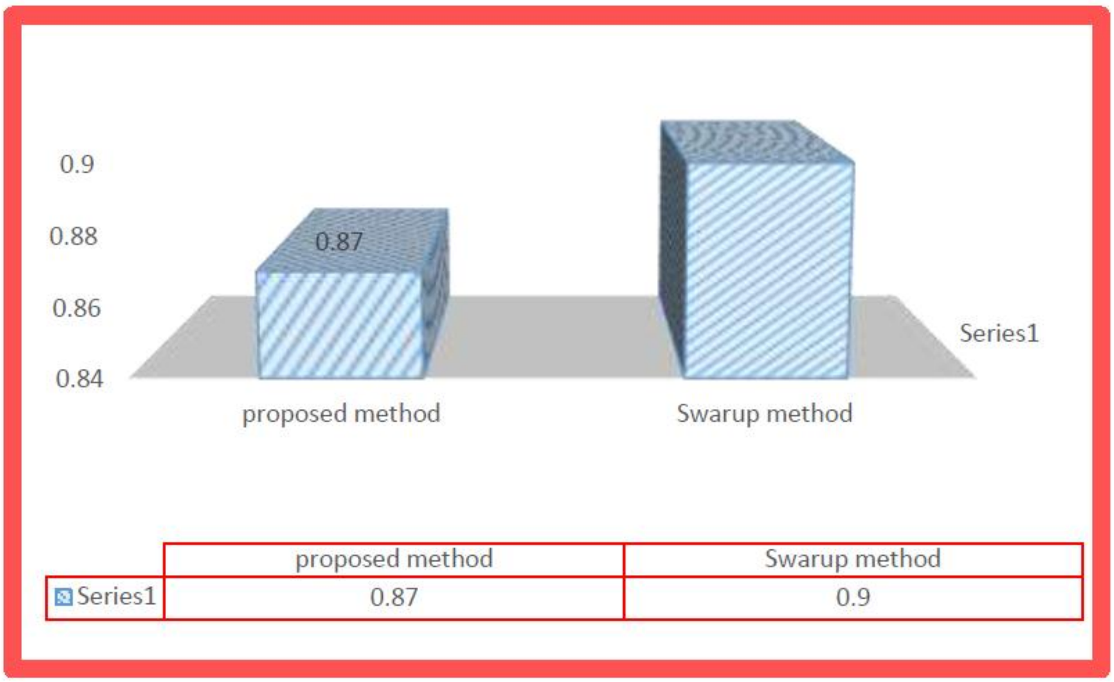

Fig: 1

Example 4.2. Solve LFP problem using HPM

$$\text{Maximize F(x)} = \frac{2x_1 + 3x_2}{x_1 + x_2 + 7} \tag{4.2.1}$$

Subject to $3x_1 + 5x_2 = 15$

$$4x_1 + 3x_2 = 12$$
$$x_1, x_2 \geq 0$$

To separate the LFP problem and written as

Maximize F(x) = $2x_1 + 3x_2$

Subject to $x_1 + x_2 + 7 = \theta$ (4.2.2)

$$3x_1 + 5x_2 = 15$$
$$4x_1 + 3x_2 = 12$$
$$x_1, x_2 \geq 0$$

Where θ is defined by Minimize F(x) = $x_1 + x_2 + 7$

Subject to
$$3x_1 + 5x_2 = 15$$
$$4x_1 + 3x_2 = 12$$
$$x_1, x_2 \geq 0$$

We define θ value

By using HPM and Factorisation method, we have

$$L = \begin{bmatrix} 1 & 0 \\ 4/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 5 \\ 0 & -11/3 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

Using (3.1.6) we obtain,

$m_1 = 0, m_2 = 10$

Thus the solution of equation (3.1.7) becomes $V = v_1 = b - 2m = [15,12]^t - [0,20]^t = [15,-8]^t$

Thus from equation (3.1.2) UX=V and solution becomes $x_1 = 1.36, x_2 = 2.18$ and Min F(x) =10.54

Hence θ=10.54 put the value in equation (4.2.1) we have,

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -1/2 & 1 \end{bmatrix}, U=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, b=\begin{bmatrix} 3.54 \\ 15 \\ 12 \end{bmatrix}$$

Using equation (3.1.6) we have,

$m_1 = 0, m_2 = 5.31, m_3 = 5.98$

Thus the solution of equation (3.1.7) becomes $[3.54,15,12]^t - [0,10.62,11.97]^t = [3.54,4.38,0.03]^t$

Thus from equation (3.1.2) the value of $x_1 = 1.35, x_2 = 2.19, x_3 = 0$ .The objective value is 0.87,

Which is the exact solution.

By comparing the result of proposed method with Swarup method, we conclude that our Method is more reliable, because:

$(Z)_{\text{proposed method}} = 0.87$

$(Z)_{\text{Swarup method}} = 0.9$

$0.87 = (Z)_{\text{proposed method}} < (Z)_{\text{Swarup method}} = 0.9$

To show the compare result of proposed method with Swarup method we also present a graphical representation:

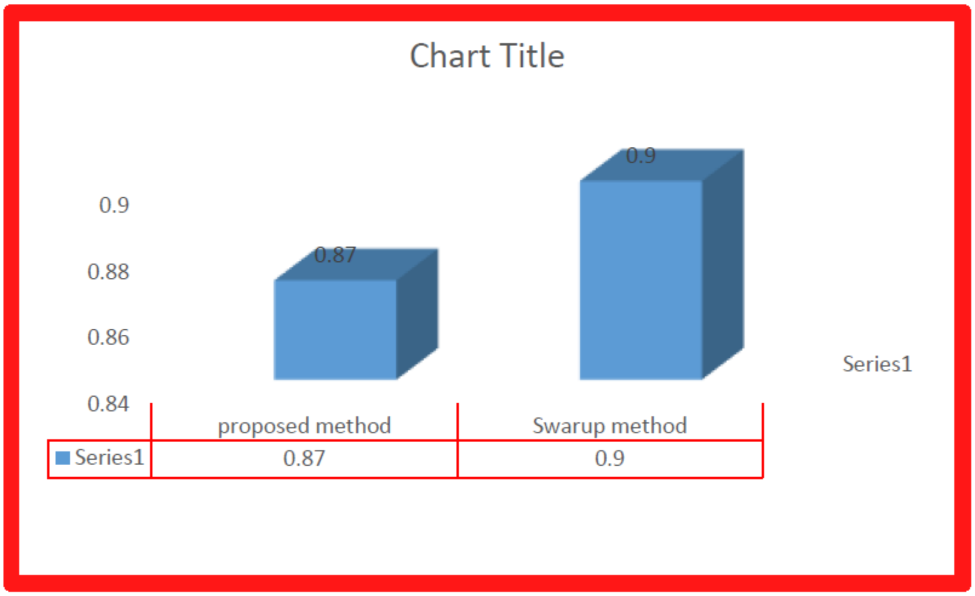

Fig: 2

5. Conclusion

In this work, in our knowledge we first proposed a new HPM method and factorisation method are introduced to solve Linear Fractional Programming (LFP) problem. A new Homotopy, H (u, p, m, m) for m vectors leads to fast convergent rate. We also present a graphical comparison result with our proposed method with existing method. By a simple example and one real life application problem, the obtained results of proposed method with swarup's method have been compared and shown the reliability and applicability of our algorithm.

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