Operations Research and Applications : An International Journal (ORAJ), Vol.2, No.1, February 2015

POINT SYMMETRIES OF LAGRANGIANS

J. López-Bonilla[1], R. López-Vázquez[1], P. Lam-Estrada[2], G. Ovando[3],

[1]ESIME-Zacatenco, Instituto Politécnico Nacional (IPN),
Edif.5, 1er. Piso, Col. Lindavista, CP 07738, México DF,
[2]Departamento de Matemáticas, ESFM-IPN, Edif. 9, Zacatenco CP 07738, México DF,
[3]CBI-Área de Física-AMA, UAM-A, Av. San Pablo 180, Col. Reynosa-Tamps.,
MéxicoDF

ABSTRACT

We give an elementary exposition of a method to obtain the infinitesimal point symmetries of Lagrangians. Besides, we exhibit the Lanczos approach to Noether's theorem to construct the first integral associated with each symmetry.

MSC 2010:*49S05, 58E30, 70H25, 70H33*

KEYWORDS

Noether's theorem, Local symmetries

1. INTRODUCTION

We consider a physical system where the parameters $q_1, q_2, \ldots, q_n$ are its generalized coordinates, that is, there are n degrees of freedom. The action:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt, \qquad \dot{q} = \frac{dq}{dt}, \tag{1}$$

is fundamental in the dynamical evolution of the system. We can change to new coordinates via the local transformations:

$$\tilde{t} = t + \varepsilon\, \alpha_0(t), \qquad \tilde{q}_i = q_i + \varepsilon\, \alpha_i(q, t), i = 1, 2, \ldots, n \tag{2}$$

where ε is an infinitesimal parameter, thus the action takes the value:

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t})\, d\tilde{t}. \tag{3}$$

If $\delta S = \delta \tilde{S}$, to first order in ε, then we say that the action is invariant under the transformations (2), that is, (2) are local symmetries of the Euler-Lagrange equations of motion.

The principal aim of this work is to show a technique to investigate the existence of point symmetries for a given action, and to realize the explicit construction of the functions α_r, $r = 0, \ldots, n$. The Sec. 2 contains the method of Emmy Noether [1-4] to achieve this aim. In Sec. 3 the Noether's theorem [5-12], in the Lanczos approach [13,14], is used to deduce the first integral associated with each point symmetry. We make applications to Lagrangians studied by several authors [4, 15-19].

2. NOETHER'S METHOD

From (2) it is easy to deduce, to first order in ε, that:

$$L\left(\tilde{q},\frac{d\tilde{q}}{d\tilde{t}},\tilde{t}\right)d\tilde{t} = L\left(q,\frac{dq}{dt},t\right)dt + \varepsilon\, N(q,\dot{q},t)dt, \tag{4}$$

with the Noether's function [4, 19]:

$$N = \frac{\partial}{\partial t}(L\alpha_0) + \frac{\partial L}{\partial q_i}\alpha_i + \frac{\partial L}{\partial \dot{q}_i}(\dot{\alpha}_i - \dot{q}_i\dot{\alpha}_0), \qquad \dot{\alpha}_i = \frac{\partial \alpha_i}{\partial t} + \frac{\partial \alpha_i}{\partial q_r}\dot{q}_r, \tag{5}$$

Where the Dedekind (1868)-Einstein [20,21] summation convention is used for repeated indices. The condition $\delta\tilde{S} = \delta S$ can be obtained if the variation of the Lagrangian is a total derivative [3,22], that is, if in (4):

$$N(q,\dot{q},t) = \frac{d}{dt}F(q,t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_a}\dot{q}_a \tag{6}$$

thus in the left side of (6) the term without velocities is equal to $\frac{\partial F}{\partial t}$, the coefficient of $\dot{q}_a$ coincides with $\frac{\partial F}{\partial q_a}$, and each term nonlinear in the $\dot{q}_j$ must be zero, which leads to a set of partial differential equations [4] called Killing equations [2] for the functions $\alpha_0(t)$ and $\alpha_r(q,t)$.

We can apply the Noether's expressions to Lagrangians employed by several authors:

a). Rothe [17].

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \tag{7}$$

then (5) and (6) imply the Killing equations:

$$\frac{\partial \alpha_1}{\partial q_2} = 0, \frac{1}{2}(q_1 - q_2)^2\dot{\alpha}_0 + q_2\frac{\partial \alpha_1}{\partial t} + (q_1 - q_2)(\alpha_1 - \alpha_2) = \frac{\partial F}{\partial t},$$
$$\frac{\partial \alpha_1}{\partial q_1} - \frac{1}{2}\dot{\alpha}_0 = 0, q_2\frac{\partial \alpha_1}{\partial q_2} = \frac{\partial F}{\partial q_2}, \alpha_2 + \frac{\partial \alpha_1}{\partial t} + q_2\frac{\partial \alpha_1}{\partial q_1} = \frac{\partial F}{\partial q_1},$$

whose general solution is:

$$\alpha_0 = c_0, \qquad \alpha_2 = \alpha_1 - \dot{\alpha}_1, \qquad F = q_1\alpha_1, \tag{8}$$

where c_0 is any constant, $\alpha_1(t)$ is an arbitrary function, and we could ask the conditions $\alpha_1(t_j) = 0$, $j = 1, 2$. Thus (2) gives the local symmetry:

$$\tilde{t} = t + \varepsilon\, c_0, \qquad \tilde{q}_1 = q_1 + \varepsilon\, \alpha_1, \qquad \tilde{q}_2 = q_2 + \varepsilon\,(\alpha_1 - \dot{\alpha}_1), \tag{9}$$

in according with the relations (2.5) in [17].
b). Henneaux [15, 17].

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 , \qquad (10)$$

and from this Noether's procedure we deduce the system of Killing equations:

$$\frac{\partial\alpha_2}{\partial q_1} = \frac{\partial\alpha_3}{\partial q_1} = 0, \frac{\partial\alpha_2}{\partial q_3} + \frac{\partial\alpha_3}{\partial q_2} = 0, -\frac{\dot{\alpha}_0}{2} + \frac{\partial\alpha_2}{\partial q_2} = 0, -\frac{\dot{\alpha}_0}{2} + \frac{\partial\alpha_3}{\partial q_3} = 0,$$

$$\frac{\partial F}{\partial t} = \frac{1}{2}(q_2^2 + e^{2q_1})\dot{\alpha}_0 - e^{q_1}\frac{\partial\alpha_2}{\partial t} - q_2\frac{\partial\alpha_3}{\partial t} + e^{2q_1}\alpha_1 + q_2\alpha_2, \frac{\partial F}{\partial q_1} = -e^{q_1}\frac{\partial\alpha_2}{\partial q_1} - q_2\frac{\partial\alpha_3}{\partial q_1},$$

$$\frac{\partial F}{\partial q_2} = \frac{\partial\alpha_2}{\partial t} - e^{q_1}\frac{\partial\alpha_2}{\partial q_2} - q_2\frac{\partial\alpha_3}{\partial q_2} - e^{q_1}\alpha_1, \frac{\partial F}{\partial q_3} = -e^{q_1}\frac{\partial\alpha_2}{\partial q_3} + \frac{\partial\alpha_3}{\partial t} - q_2\frac{\partial\alpha_3}{\partial q_3} - \alpha_2,$$

with the general solution:

$$\alpha_0 = c_0 , \qquad \alpha_1 = e^{-q_1}\ddot{\alpha}_3 , \qquad \alpha_2 = \dot{\alpha}_3 , \qquad F = 0, \qquad (11)$$

and the corresponding point symmetry has the structure:

$$\tilde{t} = t + \varepsilon\, c_0 , \qquad \tilde{q}_1 = q_1 + \varepsilon\, e^{-q_1}\ddot{\alpha}_3 , \qquad \tilde{q}_2 = q_2 + \varepsilon\, \dot{\alpha}_3 , \qquad \tilde{q}_3 = q_3 + \varepsilon\, \alpha_3 , \qquad (12)$$

where $\alpha_3(t)$ is arbitrary, which are the expressions (2.7) in [17].

c). Torres del Castillo [18].

$$L = \frac{1}{6}\dot{q}^3 + \frac{1}{2}gt\dot{q}^2 - g^2qt, g \text{ is a constant,} \qquad (13)$$

therefore (5) and (6) imply the partial differential equations:

$$-\frac{1}{3}\dot{\alpha}_0 + \frac{1}{2}\frac{\partial\alpha}{\partial q} = 0, gt\,\frac{\partial\alpha}{\partial t} = \frac{\partial F}{\partial q}, -gt\dot{\alpha}_0 + \frac{\partial\alpha}{\partial t} + 2gt\frac{\partial\alpha}{\partial q} + g\alpha_0 = 0, g^2(qt\dot{\alpha}_0 + q\alpha_0 + t\alpha) = -\frac{\partial F}{\partial t}$$

with the solution:

$$\alpha_0 = \frac{3}{2}\,t, \qquad \alpha = q - gt^2, \qquad F = g^2t^2\left(\frac{1}{4}\,gt^2 - 2q\right), \qquad (14)$$

in harmony with the transformation (2) in [18] for the infinitesimal case. Thus the local symmetry is given by:

$$\tilde{t} = t + \frac{3}{2}\,\varepsilon\, t, \qquad \tilde{q} = q + \varepsilon\,(q - gt^2). \qquad (15)$$

d). Torres del Castillo [18].

$$L = \frac{1}{2}t^2\left(\dot{q}^2 - \frac{1}{3}q^6\right), \qquad (16)$$

and this Noether's approach permits to obtain the Killing equations:

$$t^2 \frac{\partial \alpha}{\partial t} = \frac{\partial F}{\partial q}, -\frac{1}{2} t^2 \dot{\alpha}_0 + t^2 \frac{\partial \alpha}{\partial q} + t\alpha_0 = 0, t\, q^5 \left(\frac{1}{6}\, tq\dot{\alpha}_0 + \frac{1}{3}\, q\alpha_0 + t\alpha\right) = -\frac{\partial F}{\partial t},$$

with the following solution:

$$\alpha_0 = 2t, \qquad \alpha = -q, \qquad F = 0, \qquad (17)$$

equivalent to infinitesimal version of the transformation (4) in [18], and the point symmetry is:

$$\tilde{t} = t + 2\,\varepsilon t, \qquad \tilde{q} = q - \varepsilon q. \qquad (18)$$

e). Havelková [4] – Torres del Castillo [19].

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3\,, \qquad (19)$$

its associated Noether's partial differential equations system is:

$$\frac{\partial \alpha_1}{\partial q_2} = \frac{\partial \alpha_1}{\partial q_3} = \frac{\partial \alpha_3}{\partial q_1} = \frac{\partial \alpha_3}{\partial q_2} = 0, \frac{\partial \alpha_1}{\partial q_1} + \frac{\partial \alpha_3}{\partial q_3} - \dot{\alpha}_0 = 0, \frac{\partial F}{\partial q_3} = \frac{\partial \alpha_1}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial q_3} - \alpha_2\,,$$
$$\frac{\partial F}{\partial t} = q_3(q_1\dot{\alpha}_0 + \alpha_1) + q_1\alpha_3 - q_2\frac{\partial \alpha_3}{\partial t}, \frac{\partial F}{\partial q_1} = \frac{\partial \alpha_3}{\partial t} - q_2 \frac{\partial \alpha_3}{\partial q_1}, \frac{\partial F}{\partial q_2} = -q_2 \frac{\partial \alpha_3}{\partial q_2},$$

and the corresponding general solution is given in [4, 19]:

$$\alpha_0 = c_0, \qquad \alpha_1 = -c_1 q_1 + c_2 e^t + c_3 e^{-t}, \qquad \alpha_2 = -c_1 q_2 + f(q_3), \qquad \alpha_3 = c_1 q_3\,,$$
$$F = (c_2 e^t - c_3 e^{-t}) q_3 - \int^{q_3} f(u) du, \qquad (20)$$

Where $f(q_3)$ is an arbitrary function and the c_j are constants. Thus, we have the local symmetry ([4] p.28, and relations (30) in [19]):

$$\tilde{t} = t + \varepsilon\, c_0\,, \tilde{q}_1 = q_1 + \varepsilon\,(-c_1\, q_1 + c_2\, e^t + c_3 e^{-t}),$$
$$\tilde{q}_2 = q_2 + \varepsilon\left(-c_1\, q_2 + f(q_3)\right), \tilde{q}_3 = q_3 + \varepsilon\, c_1 q_3\,. \qquad (21)$$

f). Rothe [17].

$$L = \tfrac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\dot{q}_1 + \tfrac{1}{2}(q_1 - q_2 + q_3)^2, \qquad (22)$$

with the set of Killing equations:

$$\frac{\partial \alpha_1}{\partial q_2} = \frac{\partial \alpha_1}{\partial q_3} = 0, \frac{\partial \alpha_1}{\partial q_1} = \frac{1}{2}\dot{\alpha}_0\,, \frac{\partial F}{\partial q_r} = (q_2 - q_3)\frac{\partial \alpha_1}{\partial q_r}\,, \quad r = 2, 3$$
$$\frac{\partial F}{\partial t} = (q_1 - q_2 + q_3)\left[\frac{1}{2}\dot{\alpha}_0(q_1 - q_2 + q_3) + \alpha_1 - \alpha_2 + \alpha_3\right] + (q_2 - q_3)\frac{\partial \alpha_1}{\partial t},$$
$$\frac{\partial F}{\partial q_1} = \left(\frac{\partial \alpha_1}{\partial q_1} + \dot{\alpha}_0\right)(q_2 - q_3) + \frac{\partial \alpha_1}{\partial t} + \alpha_2 - \alpha_3\,,$$

which implies that $F = q_1\alpha_1$ with the point symmetry:

$\tilde{t} = t + \varepsilon\, c_0\,, \quad \tilde{q}_1 = q_1 + \varepsilon\, \alpha_1\,, \tilde{q}_2 = q_2 + \varepsilon\,(\alpha_1 - \dot{\alpha}_1 + \alpha_3), \quad \tilde{q}_3 = q_3 + \varepsilon\, \alpha_3\,,$
(23)

where $\alpha_1(t)$ and $\alpha_3(t)$ are arbitrary functions, in according with the expressions (2.36) in [17].
In this Section the Noether's technique was applied to several Lagrangians to exhibit that the explicit construction of local symmetries it is equivalent to solve a set of partial differential equations named Killing equations. Local symmetries of the action are not always easily detected; it is however crucial to unravel them since their knowledge is required for the quantization [17] of such singular systems.In Sec. 3 we use the Lanczos technique [13, 14] to deduce the conservation laws [8, 23] associated with point symmetries.

3. LANCZOS APPROACH TO NOETHER'S THEOREM

Noether [1-3, 8] proved that in a variational principle the existence of symmetries implies the presence of conservation laws. Lanczos [13, 14] employs this Noether's result in the following manner:

1). We consider a global symmetry with constant parameters.
2). After we accept that the parameters are functions of *t*, that is, now the transformation is a local symmetry.
3). Into Lagrangian we substitute this local mapping to first order in ε, and the parameters are new degrees of freedom.
4). Then the Euler-Lagrange equations for these parameters give the conservation laws.

Now we apply this Lanczos approach to Lagrangians from Sec. 2:

A). Lagrangian (7):

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2, \qquad (24)$$

First, in the transformation (9) we employ $c_0 = 0$ with the constant $\alpha_1 = a$, thus we have the global symmetry $\tilde{t} = t,\ \tilde{q}_1 = q_1 + \varepsilon\, a,\ \tilde{q}_2 = q_2 + \varepsilon\, a$. Now we change the parameter a by the function $\beta(t)$,our new degree of freedom, to obtain the local symmetry $\tilde{t} = t,\ \tilde{q}_1 = q_1 + \varepsilon\, \beta(t),\ \tilde{q}_2 = q_2 + \varepsilon\, \beta(t)$, therefore $\widetilde{L} = L + \varepsilon\left[\beta\, \dot{q}_1 + \dot{\beta}(\dot{q}_1 + q_2)\right]$, and from the Euler-Lagrange equation forβ,$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\beta}}\right) - \frac{\partial \tilde{L}}{\partial \beta} = 0$, we deduce that:

$$\frac{d}{dt}(\dot{q}_1 + q_2 - q_1) = 0, \qquad (25)$$

and it is the conserved quantity associated with (9) for $c_0 = 0$.

The transformation (9) for $\alpha_1 = 0$ is the global symmetry $\tilde{t} = t + \varepsilon\, c_0,\ \tilde{q}_1 = q_1,\ \tilde{q}_2 = q_2$, and now we consider that c_0 is the function $\beta(t)$, then $\widetilde{L}\, d\tilde{t} = \left[L + \dot{\beta}(L - \dot{q}_1^2 - \dot{q}_1 q_2)\right] dt$ and the Euler-Lagrange equation for β implies the conservation of the Hamiltonian function $H = \dot{q}_1(\dot{q}_1 + q_2) - L$, because t is ignorable in (24).

B). Lagrangian (10):

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 \,. \tag{26}$$

In the transformation (12) we utilize $c_0 = 0$ and $\alpha_3 = a =$ constant, to obtain the global symmetry $\tilde{t} = t, \tilde{q}_1 = q_1,\ \tilde{q}_2 = q_2,\ \tilde{q}_3 = q_3 + \varepsilon\, a.$ Into (25) we apply this symmetry when $a \rightarrow \beta(t)$:

$$\tilde{L} = L + \varepsilon\, \dot{\beta}\, (\dot{q}_3 - q_2) \therefore \frac{d}{dt}(\dot{q}_3 - q_2) = 0, \tag{27}$$

on the subspace of physical paths. If it is necessary, in (12) we can use $\alpha_3 = 0$ with $c_0 \neq 0$ to deduce the conservation of the corresponding Hamiltonian because (26) has not explicit dependence of *t*.

C). Lagrangian (19):

$$L = (\dot{q}_1 - q_2)\dot{q}_3 + q_1 q_3\,, \tag{28}$$

the transformations (20) permit several situations, in fact:

C1). $c_0 = c_2 = c_3 = f = 0,\ c_1 \neq 0$, then $\tilde{L} = L + \varepsilon\, \dot{\beta}\, (q_3\dot{q}_1 - q_3 q_2 - \dot{q}_3 q_1)$ and the Lanczos procedure implies:

$$\text{Constant} = q_1\, \dot{q}_3 - (\dot{q}_1 - q_2) q_3\,, \tag{29}$$

whose value is zero on-shell.

C2). $c_r = 0$, $r = 0,\ldots, 3$ and $f = a$, therefore $\tilde{t} = t,\ \tilde{q}_1 = q_1,\ \tilde{q}_2 = q_2 + \varepsilon\, a,\ \tilde{q}_3 = q_3$. If $a \rightarrow \beta(t)$, from (28) we have that $\tilde{L} = L - \varepsilon\, \beta\, \dot{q}_3$ and the Euler-Lagrange equation for β leads to:

$$\dot{q}_3 = 0, \tag{30}$$

on the subspace of physical trajectories.

C3). $c_0 = c_1 = c_3 = f = 0,\ c_2 \neq 0$, then $\tilde{L} = L + \varepsilon\left[\dot{\beta}\dot{q}_3 + \beta\,(q_3 + \dot{q}_3)\right]e^t$ and the Lanczos approach gives

$$\ddot{q}_3 - \dot{q}_3 - q_3 = 0. \tag{31}$$

D). Lagrangian (22):

$$L = \frac{1}{2}\dot{q}_1^2 + (q_2 - q_3)\dot{q}_1 + \frac{1}{2}(q_1 - q_2 + q_3)^2. \quad (32)$$

In (23) the first option is $c_0 = \alpha_3 = 0 \;\therefore \tilde{t} = t,\; \tilde{q}_1 = q_1 + \varepsilon\, a,\; \tilde{q}_2 = q_2 + \varepsilon\, a,\; \tilde{q}_3 = q_3$. If $a \rightarrow \beta(t)$,

then from (32) $\tilde{L} = L + \varepsilon\left[\dot{\beta}(\dot{q}_1 + q_2 - q_3) + \beta\, \dot{q}_1\right]$, thus:

$$\frac{d}{dt}(\dot{q}_1 + q_2 - q_3 - q_1) = 0. \quad (33)$$

The second case is $c_0 = \alpha_1 = 0,\; \tilde{t} = t,\; \tilde{q}_1 = q_1,\; \tilde{q}_2 = q_2 + \varepsilon\, a,\; \tilde{q}_3 = q_3 + \varepsilon\, a$, and if $a \rightarrow \beta(t)$ we

obtain $\tilde{L} = L \;\;\therefore$ the Euler-Lagrange equation for β gives $0 = 0$.

Our process show that with the Lanczos technique is easy to deduce the conservation laws (on the subspace on physical paths) associated with point symmetries. On this topic we recommend the interesting papers indicated in [24-27].

REFERENCES

[1] E. Noether, Invariantevariationsprobleme, Nachr. Ges.Wiss.Göttingen2 (1918) 235-257

[2] D. E. Neuenschwander, Emmy Noether's wonderful theorem, The Johns Hopkins University Press, Baltimore, USA (2011)

[3] Y. Kosmann-Schwarzbach, The Noether theorems, Springer, New York (2011)

[4] Monika Havelková, Symmetries of a dynamical system represented by singular Lagrangians, Comm. in Maths.20, No. 1 (2012) 23-32

[5] H. Rund, The Hamilton-Jacobi theory in the calculus of variations. Its role in Mathematics and Physics, D. Van Nostrand, London (1966)

[6] H. Rund, A direct approach to Noether's theorem in the calculus of variations, Utilitas Math. 2 (1972) 205-214

[7] D. Lovelock, H. Rund, Tensors, differential forms and variational principles, Dover, New York (1989)

[8] Nina Byers, EmmyNoether's discoveryof the deep connection between symmetries and conservation laws, Proc. Symp. Heritage E. Noether, Bar Ilan University, Tel Aviv, Israel, Dec. 2-3, 1996

[9] D. E. Neuenschwander, Symmetries, conservation laws and Noether's theorem, Soc. of Phys. Newsletter, Jan 1996, 14-16

[10] István Hargittai, Magdolna Hargittai, Homage to Emmy Noether, Mathematical Intelligencer 24, No. 1 (2002) 48-49

[11] K. A. Brading, H. Brown, Symmetries and Noether's theorems, in 'Symmetries in Physics, Philosophical Reflections', Cambridge University Press (2003) 89-109

[12] L. M. Lederman, Ch. T. Hill, Symmetry and the beautiful universe, Prometheus Books, Amherst, New York (2004) Chaps. 3 and 5

[13] C. Lanczos, The variational principles of mechanics, University of Toronto Press (1970) Chap. 11

[14] C. Lanczos, Emmy Noether and calculus of variations, Bull. Inst. Math. Appl. 9, No.8 (1973) 253-258

[15] M. Henneaux, C. Teitelboim, J. Zanelli, Gauge invariance and degree of freedom count, Nucl. Phys. B 332, No. 1 (1990) 169-188

[16] M. Henneaux, C. Teitelboim, Quantization of gauges systems, Princeton University Press, NJ (1994)

[17] H. J. Rothe, K. D. Rothe, Classical and quantum dynamics of constrained Hamiltonian systems, World Scientific Lecture Notes in Physics 81, Singapore (2010)
[18] G. F. Torres del Castillo, C. Andrade-Mirón, R. Bravo-Rojas, Variational symmetries of Lagrangians, Rev. Mex. Fís. E59 (2013) 140-147
[19] G. F. Torres del Castillo, Point symmetries of the Euler-Lagrange equations, Rev. Mex. Fís. 60 (2014) 129-135
[20] M. A. Sinaceur, Dedekind et le programme de Riemann, Rev. Hist. Sci. 43 (1990) 221-294
[21] D. Laugwitz, Bernhard Riemann 1826-1866. Turning points in the conception of mathematics, Birkhäuser, Boston, USA (2008)
[22] E. Bessel-Hagen, Über die Erhaltungssätze der Elektrodynamik, MathematischeAnnalen84 (1921) 258-276
[23] J. López-Bonilla, J. Morales, G. Ovando, Conservation laws for energy and momentum in curved spaces, J. Zhejiang Univ. Sci. A8, No. 4 (2007) 665-668
[24] V. N. Mishra, *Some problems on approximations of functions in Banach spaces,* Ph. D. Thesis, Indian Institute of Technology, Roorkee, Uttarakhand 247 667, India (2007)
[25] Deepmala, H. K. Pathak, *A study on some problems on existence of solutions for nonlinear functional- integral equations,* Acta Mathematica Scientia B**33**, No. 5 (2013) 1305-1313
[26] V. N. Mishra, K. Khatri, L. N. Mishra, Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators,* J. of Inequalities and Applications (2013) 2013:586, doi: 10.1186/1029-242X-2013-586
[27] Deepmala, *A study on fixed point theorems for nonlinear contractions and its applications,* Ph. D. Thesis, Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) 492 010, India (2014)