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# MEAN ABSOLUTE DEVIATION FOR HYPEREXPONENTIAL AND HYPOEXPONENTIAL DISTRIBUTIONS

Weiqi Zhang, Zibo Wang and Eugene Pinsky

Department of Computer Science, Metropolitan College, Boston University

***ABSTRACT***

*Hyperexponential and Hypoexponential distributions are derived from mixtures and convolutions of independent exponential random variables, respectively, and have a wide range of applications in telecommunications, quantitative finance, and reliability analysis. In addition, Bernstein's theorem states that all completely monotonic probability distribution functions (PDFs) can be expressed as mixtures of exponential distributions. In this paper, we not only explore these distributions but also pioneer the derivation of Mean Absolute Deviation (MAD) for them. We establish new Chebyshev-type bounds and Peek bounds that further enhance our understanding and exploitation of these distributions. Our contribution lies in providing explicit formulas for MAD calculation specific to Hyperexponential and Hypoexponential distributions and using the MAD in real-life applications.*

***KEYWORDS***

*Chebyshev's Inequality, Exponential Distribution, Probability Distributions.*

## 1. INTRODUCTION

Hyperexponential and hypoexponential distributions, as extensions of the exponential distribution, play an improtant role in capturing the intricacies of real-world processes that exhibit non-uniform rates of occurrence [1]. Bernstein's theorem states that any completely monotonic probability distribution function (PDF) can be expressed as a mixture of exponential PDFs[2]. This theory not only lays the foundation for the modeling of Hyperexponential distributions, but also reveals their high adaptability in dealing with various practical complexities (such as heavy-tailed behavior and multimodal characteristics). Hyperexponential distributions are often the case in network traffic management and other areas where service times or interarrival times are highly variable and can be better represented by a mixture of exponential distributions[3]. The heavy-tailed nature of hyperexponential distributions makes them ideal for capturing the long-tail risks and extreme events that are often critical in data-driven decision-making. Conversely, hypoexponential distributions arise in scenarios where processes consist of multiple stages, each with distinct rates, such as in multi-phase manufacturing systems or project management workflows. These distributions play an important role in modeling the cumulative effects of sequential tasks, and are particularly useful for describing situations where the total completion time of a process consists of multiple stages, where the duration of each stage is driven by its own exponential rate [4].

Despite their practical importance, the analytical complexity of hyperexponential and hypoexponential distributions has limited their widespread use in data science applications[5,6]. One of the key challenges is to derive meaningful statistical metrics, such as the mean absolute deviation (MAD), which is more accurate in assessing variability than the standard deviation [7].

Existing approaches to calculate MAD for these distributions often rely on numerical approximations or simulation methods, which can be computationally intensive and lack generalizability to different distribution parameters[8]. Moreover, these methods frequently focus on specific cases or simplified assumptions, limiting their applicability to real-world scenarios involving complex and dynamic systems.

In this paper, we bridge this gap by developing explicit formulas for calculating MAD for both hyperexponential and hypoexponential distributions. Additionally, we introduce new Chebyshev-type bounds and Peek bounds based on MAD, offering enhanced tools for the analysis of these distributions in practical applications. By applying these methods to real data, we verify their effectiveness in capturing the variability and complexity inherent in modern data-driven systems. These works not only deepen our understanding of MAD theory, but also provide practitioners with powerful tools in decision-making and operational optimization.

## 2. MEAN ABSOLUTE DEVIATIONS VIA CUMULATIVE DISTRIBUTION FUNCTIONS

Consider a real-valued random variable X on a sample space $\Omega \subseteq R$ with density f(x) and cumulative distribution function F(x). Let μ denote the mean E(X), M denote the median of X and σ denote its standard deviation. The mean absolute deviation of X from μ as [9]

$$\mathrm{H} = \int_{\Omega} |x - \mu| \, d\,F(x) \tag{2.1}$$

This is defined in the sense of Lebesque-Stieltjes integration and applies to continuous and discrete distributions.

In this paper, we compute MAD for hyper- and hypo-exponential distributions directly from cumulative distribution functions.

We will find it convenient to introduce the following auxiliary integral

$$I(z) = \int_{x \le z} x\, dF(x) \tag{2.2}$$

We can consider I(z) as a partial mean of X computed over all $x \le z$. We can express H(X,a) in terms of the cumulative distribution function F(·) and auxiliary integral I(·) as follows

$$\begin{aligned} H(X,a) &= \int_{x \le a} (a - x)\, dF(x) \; + \int_{x > a} (x - a)\, dF(x) \\ &= \left( a \int_{x \le a} dF(x) - \int_{x > a} x\, dF(x) \right) \\ &+ \left( \int_{x > a} dF(x) - a \int_{x > a} dF(x) \right) \\ &= (aF(a) - I(a)) \; + \; (\mu - I(a) - a(1 - F(a)\,) \\ &= a(2F(a) - 1) + \mu - 2I(a) \end{aligned} \tag{2.3}$$

Consider the function h(x) = xF(x). Then dh(x) = F(x)dx+xdF(x) and applying the integration by parts formula, we can rewrite I(a) as

$$I(a) = \int_{x \leq a} x\, dF(x) = aF(a) - \int_{x \leq a} F(x)\, dx \tag{2.4}$$

Substituting this expression into equation (2.3), we obtain

$$H(X, a) = (\mu - a) + 2\int_{x \leq a} F(x)\, dx \tag{2.5}$$

For MAD (around mean), where a = μ, this yields the expression for the first absolute central moment of X:

$$H(X, \mu) = 2\int_{x \leq \mu} F(x)\, dx \tag{2.6}$$

As a simple example, suppose X is distributed according to an exponential distribution with rate λ > 0 [10, 11]. The probability density function f(x) and its cumulative distribution function F(x) of the distribution are given by

$$f(x) = \lambda e^{-\lambda x} \text{ and } F(x) = 1 - e^{-\lambda x}, x \geq 0 \tag{2.7}$$

For this distribution, μ = 1/λ and σ = 1/λ. We compute H as follows:

$$\begin{aligned} H(X, \mu) &= 2\int_{x \leq \mu} F(x)\, dx = 2\int_{0}^{1/\lambda} (1 - e^{-\lambda x})\, dx \\ &= 2\left(x + \frac{e^{-\lambda x}}{\lambda}\right)\Bigg|_{0}^{1/\lambda} = \frac{2}{\lambda e} \end{aligned} \tag{2.8}$$

Expressed in terms of σ, this distribution yields: H = (2/e) σ ≈ 0.74σ

## 3. Geometric Interpretation of MAD via ”folded” Cumulative Distribution Functions

We can interpret H(x,a) as follows. Define the function G(x) as:

$$G(x, a) = \begin{cases} F(x) & x \in \Omega_L(a) \\ 1 - F(x) & x \in \Omega_R(a) \\ 0 & otherwise \end{cases} \tag{3.1}$$

We can interpret G(x) as the “folded” cumulative distribution function curve. Note that unless F(a) = 1/2 (i.e., a is the median M), the curve G(x) is discontinuous at x = a. Figure 1 illustrates the special case where a = μ.

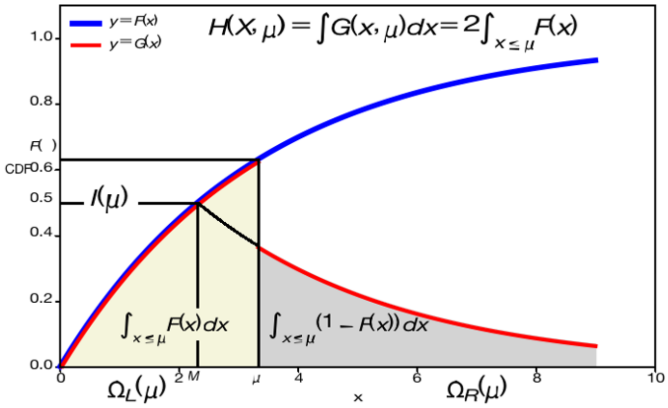



Figure 1: Illustration of Folded CDF (a = μ).

Let us compute the area A(G) under this curve. In the continuous case, we have

$$\begin{aligned} A(G) &= \int_{\Omega} G(x,a)dx \\ &= \int_{x\le a} F(x)dx + \int_{x>a} \big(1-F(x)\big)dx \\ &= \int_{x\le a}\left(\int_{y\le x} dF(y)\right)dx + \int_{x>a}\left(\int_{y>x} dF(y)\right)dx \\ &= \int_{y\le a}\left(\int_{y}^{a} dx\right)dF(y) + \int_{y>a}\left(\int_{a}^{y} dx\right)dF(y) \qquad (3.2) \\ &= \int_{y\le a} (a-y)dF(y) + \int_{y>a} (y-a)dF(y) \\ &= \int_{\Omega} |a - y|\, dF(y) \; = \; H(X,a) \end{aligned}$$

We can interpret $A_L = aF(a) - I(a)$ as the area under the left part of curve of function G(x,a) when $x \le a$ and $A_R = (\mu - I(a) - a(1 - F(a))$ as the area under the right side when $x > a$.

When $a = \mu$, the areas on the left and right are equal. In this case, $H(X, \mu) = 2A_L(G)$, which is exactly twice the area of the left part of the folded cumulative distribution function (CDF) curve when $x \le a$, as shown in Figure 1. According to formula (2.4), the auxiliary integral $I(\mu)$ represents the area under the curve of the cumulative distribution function F(x) under the conditions $x \le \mu$ and $F(x) \le F(\mu)$. For the case $a = M$, the area under the curve is the mean absolute deviation from the median.

## 4. Hyperexponential Distribution

The hyperexponential distribution arises as a convolution of n independent exponential distributions each with its own rate parameter $\lambda_i$ [12], corresponding to the rate of the ith exponential component. It belongs to the broader class of phase-type distributions [13]. Consider k independently distributed exponential random variables $X_i$ each with its rate $\lambda_i > 0$. Let fi(x) and Fi(x) denote the density and cumulative distribution function of XI. Let pi denote the probability that a random variable X follows an exponential distribution with rate $\lambda_i$.

Then the density and cumulative distribution function of x are given by

$$f(x) = \sum_{i=1}^{k} p_i f_i(x) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i x} \tag{4.1}$$

$$F(x) = \sum_{i=1}^{k} p_i F_i(x) = \sum_{i=1}^{k} p_i \left(1 - e^{-\lambda_i x}\right) \tag{4.2}$$

The mean and variance of X are given by

$$\mu = \frac{p_1}{\lambda_1} + \cdots + \frac{p_k}{\lambda_k} \tag{4.3}$$

$$\sigma^2 = 2\left(\frac{p_1}{\lambda_1^2} + \cdots + \frac{p_k}{\lambda_k^2}\right) - \left(\frac{p_1}{\lambda_1} + \cdots + \frac{p_k}{\lambda_k}\right)^2 \tag{4.4}$$

The mean absolute deviation

$$\begin{aligned} H &= 2\int_{x \le \mu} F(x) = 2\int_{x \le \mu} \left[\sum_{i=1}^{k} p_i (1 - e^{-\lambda_i x})\right] dx \\ &= 2\sum_{i=1}^{k} p_i \left[x + \frac{e^{-\lambda_i x}}{\lambda_i}\right] \Bigg|_0^{\mu} \\ &= 2\sum_{i=1}^{k} p_i \left[\mu + \frac{1}{\lambda_i}\left(e^{-\lambda_i \mu} - 1\right)\right] = 2\sum_{i=1}^{k} p_i \frac{e^{-\lambda_i \mu}}{\lambda_i} \end{aligned} \tag{4.5}$$

We note that standard deviation σ and MAD H require O(k) operations.

**Example 1:** Suppose k = 2 with rates p = p1 and q = p2 = 1 − p. Then the density and cumulative distribution functions are

$$f(x) = p_1 \lambda_1 e^{-\lambda_1 x} + p_2 \lambda_2 e^{-\lambda_2 x} \tag{4.6}$$

$$F(x) = p_1(1 - e^{-\lambda_1 x}) + p_2(1 - e^{-\lambda_2 x}) \tag{4.7}$$

The mean and the variance are

$$\mu = \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2} \tag{4.8}$$

$$\sigma^2 = 2\left(\frac{p_1}{\lambda_1^2} + \frac{p_2}{\lambda_2^2}\right) - \left(\frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2}\right)^2 \tag{4.9}$$

The mean absolute deviation is

$$H = 2p_1 \frac{e^{-\lambda_1 \mu}}{\lambda_1} + 2p_2 \frac{e^{-\lambda_2 \mu}}{\lambda_2} \tag{4.10}$$

**Example 2:** Suppose k = 3 with rates $p_1$, $p_2$, and $p_3$. Then the density and cumulative distribution functions are

$$f(x) = p_1 \lambda_1 e^{-\lambda_1 x} + p_2 \lambda_2 e^{-\lambda_2 x} + p_3 \lambda_3 e^{-\lambda_3 x} \tag{4.11}$$

$$F(x) = p_1(1 - e^{-\lambda_1 x}) + p_2(1 - e^{-\lambda_2 x}) + p_3(1 - e^{-\lambda_3 x}). \tag{4.12}$$

The mean and the variance are

$$\mu = \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2} + \frac{p_3}{\lambda_3}, \tag{4.13}$$

$$\sigma^2 = 2\left(\frac{p_1}{\lambda_1^2} + \frac{p_2}{\lambda_2^2} + \frac{p_3}{\lambda_3^2}\right) - \left(\frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2} + \frac{p_3}{\lambda_3}\right)^2 \tag{4.14}$$

The mean absolute deviation is

$$H = 2\left(p_1 \frac{e^{-\lambda_1 \mu}}{\lambda_1} + p_2 \frac{e^{-\lambda_2 \mu}}{\lambda_2} + p_3 \frac{e^{-\lambda_3 \mu}}{\lambda_3}\right) \tag{4.15}$$

## 5. HYPOEXPONENTIAL DITRIBUTION

The Hypoexponential distribution is the distribution of the sum of n ≥ 2 independent exponential random variables, each with a distinct rate parameter [14]. Consider k independently distributed exponential random variables Xi, each with its rate $\lambda_i > 0$. The density of Xi is fi(x) = $\lambda_i$ $e^{-x\lambda_i}$ , and the corresponding cumulative distribution function is Fi(x) = 1 − $e^{-x\lambda_i}$.

Consider the random variable X = X1 + ⋯ + Xk. Then X has a hypoexponential distribution. Its density can be written as

$$f(x) = \sum_{i=1}^{k} \left( \prod_{j=1, j\neq i}^{k} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) f_i(x) = \sum_{i=1}^{k} L_i(0)\lambda_i e^{-x\lambda_i} \tag{5.1}$$

where $L_1(x),...,L_k(x)$ denote the Lagrange basis polynomials defined by

$$L_i(x) = \prod_{j=1, j\neq i}^{k} \frac{x - \lambda_j}{\lambda_i - \lambda_j} \tag{5.2}$$

For notational convenience, let $l_i = L_i(0)$. The coefficients li can be written explicitly as

$$l_i = \begin{cases} \prod_{j=1, j\neq i}^{k} \frac{-\lambda_j}{\lambda_i - \lambda_j}, & \text{if } k \text{ is odd,} \\ -\prod_{j=1, j\neq i}^{k} \frac{-\lambda_j}{\lambda_i - \lambda_j}, & \text{if } k \text{ is even.} \end{cases} \tag{5.3}$$

The sum of these coefficients satisfies the property

$$l_1 + l_2 + \cdots + l_k = 1, \tag{5.4}$$

which follows from the fundamental property of Lagrange basis polynomials:

$$L_1(x) + L_2(x) + \cdots + L_k(x) = 1. \tag{5.5}$$

mean and the variance of this distribution are

$$\mu = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_k}, \sigma^2 = \frac{1}{\lambda_1^2} + \cdots + \frac{1}{\lambda_k^2}. \tag{5.6}$$

The cumulative distribution function (CDF) can be reformulated as

$$F(x) = 1 - \sum_{i=1}^{k} l_i e^{-x\lambda_i} = \sum_{i=1}^{k} l_i \left[1 - e^{-x\lambda_i}\right] = \sum_{i=1}^{k} l_i F_i(x). \tag{5.7}$$

The mean absolute deviation from equation (2.6) is

$$\begin{aligned} H &= 2\int_{x=0}^{\mu} F(x)\,dx = 2\int_{x=0}^{\mu} \left[\sum_{i=1}^{k} l_i\left[1 - e^{-x\lambda_i}\right]\right] dx \\ &= 2\sum_{i=1}^{k} l_i \left[\int_{x \le \mu} \left[1 - e^{-x\lambda_i}\right] dx\right] = 2\sum_{i=1}^{k} l_i \left[x + \frac{1}{\lambda_i} e^{-x\lambda_i}\right] \Bigg|_0^{\mu} \\ &= 2\sum_{i=1}^{k} l_i \left[\mu + \frac{1}{\lambda_i}\left(e^{-\lambda_i \mu} - 1\right)\right] = 2\sum_{i=1}^{k} l_i \frac{e^{-\lambda_i \mu}}{\lambda_i}. \end{aligned} \tag{5.8}$$

We note that the expressions for mean absolute deviations for MAD for hyperexponential distribution in equation (4.5) and the expression for MAD for hypoexponential distribution in equation (5.8) have the same form under the correspondence: $p_i \Longleftrightarrow l_i$

$$\begin{aligned} &\text{Hyperexponential: } H = 2\sum_{i=1}^{k} p_i \frac{e^{-\lambda_i \mu}}{\lambda_i} \\ &\text{Hypoexponential: } H = 2\sum_{i=1}^{k} l_i \frac{e^{-\lambda_i \mu}}{\lambda_i} \end{aligned} \tag{5.9}$$

If we define $H_i = 2/e\lambda_i$ as the mean absolute deviation for $X_i$ then we can re-write the above expressions as

$$\begin{aligned} &\text{Hyperexponential: } H = \sum_{i=1}^{k} \left(p_i\, e^{1-\lambda_i \mu}\right) H_i \\ &\text{Hypoexponential: } H = \sum_{i=1}^{k} \left(l_i\, e^{1-\lambda_i \mu}\right) H_i \end{aligned} \tag{5.10}$$

For both distributions, the mean absolute deviation for H is the appropriate weighted average of $H_i, \ldots, H_k$. In the computation of MAD for both distributions, we sum k terms. The computation of each term requires O(k) operations. This gives us $O(k^2)$ complexity to compute H.

**Example 1:** Suppose k = 2. Then we have

$$l_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} \text{ and } l_2 = \frac{-\lambda_1}{\lambda_2 - \lambda_1} \tag{5.11}$$

Then the density function is given by

$$f(x) = l_1\lambda_1\, e^{-x\lambda_1} + l_2\lambda_2\, e^{-x\lambda_2} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left(e^{-x\lambda_2} - e^{-x\lambda_1}\right) \tag{5.12}$$

The cumulative distribution function is

$$F(x) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 x} \tag{5.13}$$

The mean μ and variance $\sigma^2$ are given by

$$\mu = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \quad \text{and} \quad \sigma^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \tag{5.14}$$

Therefore, the mean absolute deviation

$$\begin{aligned} H &= \left(l_1 e^{1-\lambda_1\mu}\right)H_1 + \left(l_2 e^{1-\lambda_2\mu}\right)H_2 \\ &= \frac{\lambda_2 e^{-\lambda_1/\lambda_2}}{(\lambda_2 - \lambda_1)}\, H_1 - \frac{\lambda_1 e^{-\lambda_2/\lambda_1}}{(\lambda_2 - \lambda_1)}\, H_2 \end{aligned} \tag{5.15}$$

**Example 2:** Suppose k = 3. Then we have

$$l_1 = \frac{\lambda_2\lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}, \tag{5.16}$$

$$l_2 = \frac{\lambda_1\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}, \tag{5.17}$$

$$l_3 = \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \tag{5.18}$$

Then the density function is given by

$$\begin{aligned} f(x) &= l_1\lambda_1\, e^{-x\lambda_1} + l_2\lambda_2\, e^{-x\lambda_2} + l_3\lambda_3\, e^{-x\lambda_3} \\ &= \frac{\lambda_1\lambda_2\lambda_3\, e^{-x\lambda_1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{\lambda_1\lambda_2\lambda_3\, e^{-x\lambda_2}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \\ &+ \frac{\lambda_1\lambda_2\lambda_3\, e^{-x\lambda_3}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \end{aligned} \tag{5.19}$$

The cumulative distribution function is

$$\begin{aligned} F(x) &= l_1(1 - e^{-x\lambda_1}) + l_2(1 - e^{-x\lambda_2}) + l_3(1 - e^{-x\lambda_2}) \\ &= \frac{\lambda_2\lambda_3(1 - e^{-x\lambda_1})}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{\lambda_1\lambda_3(1 - e^{-x\lambda_2})}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \\ &+ \frac{\lambda_1\lambda_2(1 - e^{-x\lambda_3})}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \end{aligned} \tag{5.20}$$

The mean μ and variance $\sigma^2$ are given by

$$\mu = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \text{ and } \sigma^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \tag{5.21}$$

The mean absolute deviation

$$\begin{aligned} H &= \left(l_1 e^{1-\lambda_1 \mu}\right) H_1 + \left(l_2 e^{1-\lambda_2 \mu}\right) H_2 + \left(l_3 e^{1-\lambda_3 \mu}\right) H_3 \\ &= \left(\frac{\lambda_2 \lambda_3 \, e^{-(\lambda_1/\lambda_2 + \lambda_1/\lambda_3)}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}\right) H_1 \\ &+ \left(\frac{\lambda_1 \lambda_3 \, e^{-(\lambda_2/\lambda_1 + \lambda_2/\lambda_3)}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}\right) H_2 \\ &+ \left(\frac{\lambda_1 \lambda_2 \, e^{-(\lambda_3/\lambda_1 + \lambda_3/\lambda_2)}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right) H_3 \end{aligned} \tag{5.22}$$

# 6. CHEBYSHEV'S INEQUALITY, MAD-BASED INEQUALITIES AND PEEK BOUND ON MEAN ABSOLUTE DEVIATIONS

Chebyshev's inequality provides a fundamental upper bound on the probability that a random variable deviates from its mean by a certain amount [15]. However, it has limitations, especially when dealing with nonnormal distributions or outliers. This has led to the development of more specific bounds such as MAD-based inequalities and the Peek bound, which offer tailored approaches to measuring deviations. By comparing these different bounds, we can make targeted choices based on the specific characteristics of the distribution and thus more accurately assess the probability of deviations from the mean.

**Part 1:** Chebyshev's inequality gives an upper bound on the probability that a random variable X will deviate from its mean μ by more than a certain distance.

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \tag{6.1}$$

This inequality is useful for k ≥ 1. This inequality follows from the so-called Pearson inequality with r = 2 [16].

$$P\left|X - \mu\right| \geq kV_r^{1/r}\right) \leq \frac{1}{k^2}, \text{ where } V_r = \mathrm{E}(|X - \mu|^r) \tag{6.2}$$

For r = 1, there is a lesser-known inequality that gives a bound relative to the mean in terms of the mean absolute deviation H

$$P(|X - \mu| \geq kH) \leq \frac{1}{k} \tag{6.3}$$

We compare the Chebyshev inequality in (6.1) with the inequality based on mean absolute deviation (MAD) in (6.3). The MAD-based H inequality in (6.3) can be rewritten in terms of standard deviation σ as follows

$$P(|X - \mu| \geq k\sigma) = P\left(|X - \mu| \geq \frac{k\sigma}{H} \cdot H\right) \leq \frac{H}{k\sigma} \tag{6.4}$$

Comparing equations (6.1) and (6.4) we find that for MAD-based upper bound for $1 \le k \le \sigma/H$, the upper bound of H based on the mean absolute deviation (MAD) is lower than the upper bound of Chebyshev's inequality [17].

**Part 2:** Peek Bound and MAD-based Inequalities. The Peek inequality is used to give an upper bound on the probability of a random variable deviating from its expected value. Its form is as follows [18]:

$$P(|X-\mu| \ge k\sigma) \le \frac{1-(H/\sigma)^2}{k^2-2k(H/\sigma)+1}, k > H/\sigma \tag{6.5}$$

This inequality means that the further the random variable X deviates from its expected value μ, the smaller the probability of its occurrence. The upper bound of this probability is controlled by k and H/σ, where H/σ represents the deviation of the standardized random variable from its expected value, and k represents the multiple of the deviation. The Peek inequality has important applications in statistical inference and probability analysis, especially in controlling the probability of a random variable deviating from its expected value. Peek bounds based on the mean absolute deviation (MAD) can achieve a robust deviation measure, which is particularly suitable for dealing with data situations where there may be outliers or non-normal distributions.

**Example 1:** If X is exponentially distributed with rate λ, then

$$\begin{aligned} P(|X-\mu| \ge k\sigma) &= P\left(\left|X-\frac{1}{\lambda}\right| \ge \frac{k}{\lambda}\right) \\ &= P\left(X \le \frac{k-1}{\lambda}\right) + P\left(X \ge \frac{k+1}{\lambda}\right) \\ &= F\left(\frac{k-1}{\lambda}\right) + 1 - F\left(\frac{k+1}{\lambda}\right) \\ &= 1 + e^{-(k+1)} - e^{-(k-1)} \\ &= 1 - e^{-k}\left(e - \frac{1}{e}\right) \end{aligned} \tag{6.6}$$

Chebyshev and MAD-based bounds are

$$P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2} \tag{6.7}$$

$$P(|X-\mu| \ge k\sigma) \le \frac{2}{ke} \tag{6.8}$$

This must be redone for exponential: Comparing Chebyshev and MAD-based bounds in equation (6.8) we find that for $k < e/2 \approx 1.36$, MAD-based bound is better, whereas for $k > e/2$, the Chebyshev bound is better. For the Peek bound, we have $H/\sigma = 2/e$. it is easy to show that this bound coincides with Chebyshev and MAD-based bound for $k = e/2$. For $k > e/2$ this bound gives the best value that is shown in Fig. 2, MAD based bounds are effective when $k < 1.36$, indicating they are well-suited for capturing small deviations in real-world scenarios where variability is moderate and outliers are minimal. However, for $k > 1.36$, the Peek bound performs better, making it ideal for cases with large deviations or extreme variability, such as rare events or heavy-tailed distributions. This means in practical applications, the choice of bounds should depend on the extent of deviation being analysed: MAD-based for small deviations and Peek for large, ensuring more accurate and context-sensitive probability estimates.

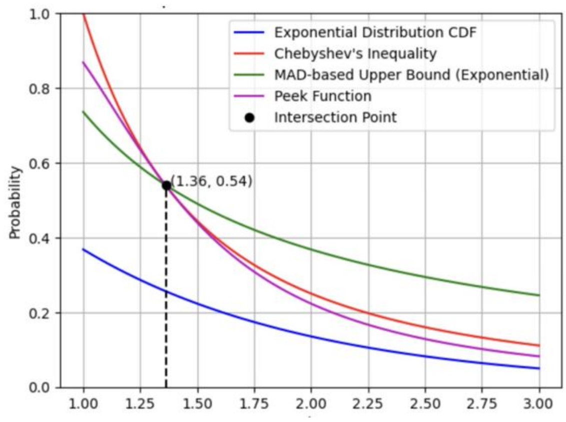



Figure 2: Exponential Distribution CDF and Bonds

In Table 1, for k = 2 from equation (6.5) we obtain for the Peek bound:

$$P(|X-\mu| \geq 2\sigma) \leq \frac{1-(2/e)^2}{4-4(2/e)+1} \approx 0.22 \tag{6.9}$$

This gives a marginally better bound than Chebyshev (0.25) and much better than MAD-based H/kσ = 1/e ≈ 0.37.

Table 1: Comparison of Bounds and Exponential Distribution CDF

| k | Chebyshev Bound | MAD-based Bound | Peek Bound | Exp Dist CDF |
|---|---|---|---|---|
| 1.0 | 1.00 | 0.74 | 0.87 | 0.37 |
| 1.5 | 0.44 | 0.49 | 0.44 | 0.22 |
| 2.0 | 0.25 | 0.37 | 0.22 | 0.14 |
| 2.5 | 0.16 | 0.29 | 0.13 | 0.08 |
| 3.0 | 0.11 | 0.25 | 0.08 | 0.05 |
| 3.5 | 0.08 | 0.21 | 0.06 | 0.03 |
| 4.0 | 0.06 | 0.18 | 0.04 | 0.02 |

**Example 2:** For Hyperexponential distribution the Chebyshev is:

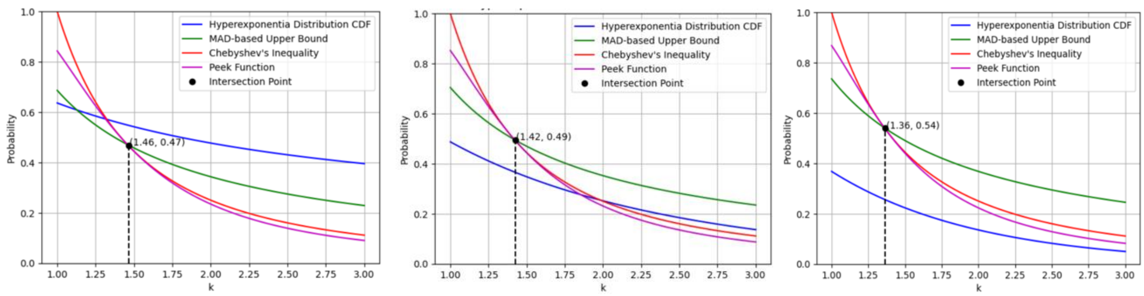
$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \tag{6.10}$$

Suppose k = 2, then the MAD-based is:

$$P(|X-\mu| \geq k\sigma) \leq \frac{2 \cdot p_1 \cdot \left(\frac{e^{-\lambda_1 \cdot \mu}}{\lambda_1}\right) + 2 \cdot p_2 \cdot \left(\frac{e^{-\lambda_2 \cdot \mu}}{\lambda_2}\right)}{\sqrt{2 \cdot \left(\frac{p_1}{\lambda_1^2} + \frac{p_2}{\lambda_2^2}\right) - \left(\frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2}\right)^2} \cdot k} \tag{6.11}$$

The Peek bound MAD-based is:

$$P(|X-\mu| \geq k\sigma) \leq \frac{1-\left(\frac{2\cdot p_1\cdot\left(\frac{e^{-\lambda_1\cdot\mu}}{\lambda_1}\right)+2\cdot p_2\cdot\left(\frac{e^{-\lambda_2\cdot\mu}}{\lambda_2}\right)}{\sqrt{2\cdot\left(\frac{p_1}{\lambda_1^2}+\frac{p_2}{\lambda_2^2}\right)-\left(\frac{p_1}{\lambda_1}+\frac{p_2}{\lambda_2}\right)^2}}\right)^2}{k^2-2k\left(\frac{2\cdot p_1\cdot\left(\frac{e^{-\lambda_1\cdot\mu}}{\lambda_1}\right)+2\cdot p_2\cdot\left(\frac{e^{-\lambda_2\cdot\mu}}{\lambda_2}\right)}{\sqrt{2\cdot\left(\frac{p_1}{\lambda_1^2}+\frac{p_2}{\lambda_2^2}\right)-\left(\frac{p_1}{\lambda_1}+\frac{p_2}{\lambda_2}\right)^2}}\right)+1} \tag{6.12}$$




(a) $\lambda_1 = 1$ and $\lambda_2 = 0.1$ (b) $\lambda_1 = 1$ and $\lambda_2 = 0.5$ (c) $\lambda_1 = 1$ and $\lambda_2 = 1$

Figure 3: Variation of Hyperexponential CDF and Bonds with $\lambda_1$ and $\lambda_2$.

In Fig. 3, we observe that with $\lambda_1$ fixed, as $\lambda_2$ decreases, the intersection points of the MAD-based Peek and Chebyshev's bounds shift leftward, indicating a decrease in the deviation threshold where MAD-based bounds outperform Chebyshev's. Additionally, the exact line moves closer to the x-axis, reflecting an overall reduction in probability values for larger deviations. This behaviour highlights the impact of $\lambda_2$ on the variability captured by the hyperexponential distribution: as $\lambda_2$ decreases, the distribution skews further, resulting in tighter bounds for smaller deviations and more pronounced long-tail behaviour.

**Example 3:** For Hypoexponential Distribution the Chebyshev is:

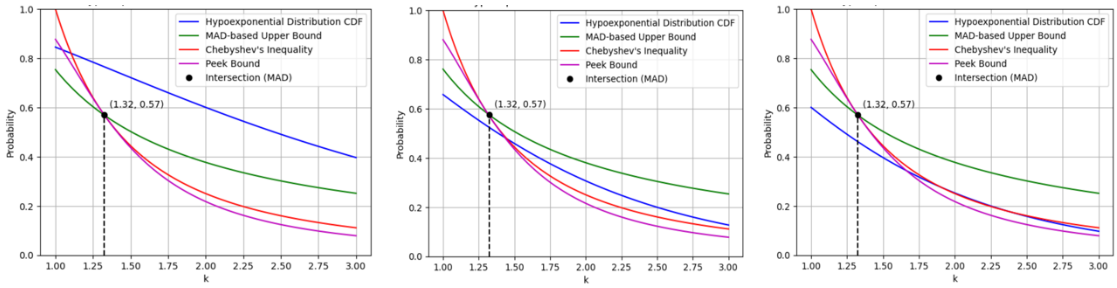
$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \tag{6.13}$$

Suppose k = 2, then the MAD-based is:

$$P(|X-\mu| \geq k\sigma) \leq \frac{2\cdot\left(\frac{\lambda_1}{\lambda_2}\cdot e^{-\lambda_2\cdot\mu}-\frac{\lambda_2}{\lambda_1}\cdot e^{-\lambda_1\cdot\mu}\right)}{\lambda_2-\lambda_1}\cdot\frac{1}{\sqrt{\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}}}\cdot\frac{1}{k} \tag{6.14}$$

The Peek bound MAD-based is:

$$P(|X-\mu| \geq k\sigma) \leq \frac{1-\left(\dfrac{2\left(\dfrac{l_1 e^{-\lambda_1 \mu}}{\lambda_1}+\dfrac{l_2 e^{-\lambda_2 \mu}}{\lambda_2}\right)}{\sqrt{\dfrac{1}{\lambda_1^2}+\dfrac{1}{\lambda_2^2}}}\right)^2}{k^2-2k\dfrac{2\left(\dfrac{l_1 e^{-\lambda_1 \mu}}{\lambda_1}+\dfrac{l_2 e^{-\lambda_2 \mu}}{\lambda_2}\right)}{\sqrt{\dfrac{1}{\lambda_1^2}+\dfrac{1}{\lambda_2^2}}}+1} \tag{6.15}$$




(a) $\lambda_1 = 1$ and $\lambda_2 = 0.5$ (b) $\lambda_1 = 1$ and $\lambda_2 = 1.5$ (c) $\lambda_1 = 1$ and $\lambda_2 = 2$

Figure 4: Variation of Hypoexponential CDF and Bonds with $\lambda_1$ and $\lambda_2$.

In Fig. 4, when $\lambda_1$ and $\lambda_2$ are the same as those in the Hyperexponential Distribution, the exact line of the Hypoexponential Distribution tends to be farther from the x-axis, indicating higher probabilities for the same deviation threshold k. This occurs because the Hypoexponential Distribution represents the sum of independent exponential variables, leading to a more spread-out distribution with reduced variability compared to the Hyperexponential Distribution, which models a mixture of exponential variables. As a result, for the same parameter values, the Hypoexponential Distribution exhibits a slower decay of probabilities, emphasizing its applicability to systems with cumulative processes, such as multi-stage operations or queues. This difference underscores how the structural characteristics of these distributions affect their behaviour and the bounds applied to them.

## 7. Case Study for Hyperexponential and Hypoexponential Distribution

We often use hyperexponential and hypoexponential distributions to analyze queueing problems [19, 20]. To overcome the limitations of a single exponential distribution in queuing systems, standard models usually adopt a mixture of exponential distributions rather than directly deriving a new distribution model suitable for specific demands [21].

We have utilized data from a call center to conduct a detailed analysis. This dataset primarily details operational metrics from a series of days for a call center operational from 8:00 AM to 6:00 PM, Monday to Friday.

Table 2: Call Data Sample

| Incoming Calls | Answered Calls | Answer Rate | Waiting Time | Talk Duration | Service Level |
|---|---|---|---|---|---|
| 217 | 204 | 94.0% | 02:45 | 02:14 | 76.3% |
| 200 | 182 | 91.0% | 06:55 | 02:22 | 72.7% |
| 216 | 198 | 91.7% | 03:50 | 02:38 | 74.3% |
| 155 | 145 | 93.6% | 03:12 | 02:29 | 79.6% |

We can see the data sample in Table 2. For convenience in calculations, we will convert "Waiting Time (AVG)" and "Talk Duration Second (AVG)" into seconds.

Consider the system model shown in Figure 5. We choose hypoexponential distribution for waiting time data and hyperexponential distribution for talk duration data. Hypoexponential distributions suit systems with multiple phases of varying rates, akin to series models, observed in scenarios like call centers. Hyperexponential distributions, akin to parallel models, capture variability in service times common in tasks with diverse durations. It's worth noting that hypoexponential distributions are akin to series models, while hyperexponential distributions are akin to parallel models [12].

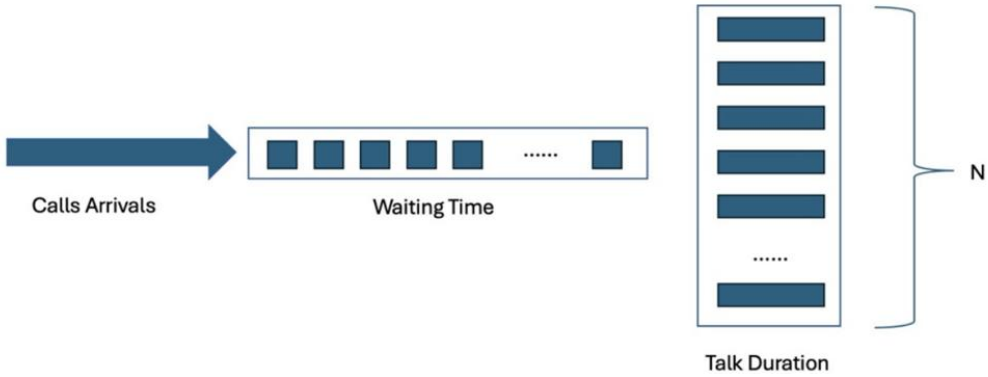



Figure 5: Structure of the Call Center model

**Case 1: Modeling Talk Duration with Hyperexponential Distribution**

We attempt to fit the talk duration time data using a hyperexponential distribution model. The hyperexponential distribution is often used to model scenarios where the coefficient of variation $c_{2s} > 1$ because it is more analytic than the Pareto distribution or other heavy-tailed distributions [22].

The mean and variance of X are given by

$$\mu = \frac{p_1}{\lambda_1} + \cdots + \frac{p_k}{\lambda_k} \tag{7.1}$$

$$\sigma^2 = 2\left(\frac{p_1}{\lambda_1^2} + \cdots + \frac{p_k}{\lambda_k^2}\right) - \left(\frac{p_1}{\lambda_1} + \cdots + \frac{p_k}{\lambda_k}\right)^2 \tag{7.2}$$

We know the $\lambda_i$ and pi for exponential distribution

$$\lambda_k = \frac{N_k}{\sum_{i=1}^{n} x_i} \quad \text{and} \quad p_k = \frac{N_k}{N} \tag{7.3}$$

where $N_k$ is the number of observations in group k, xi is the i-th observation. And $N_k$ is the number of observations in group k and N is the total number of observations.

We group the "Talk Duration Second (AVG)", dividing them into k = [2,8,20,40,60,100], and then calculate according to the formula. Finally, we obtain the Table 3. From this table, we can observe that the means are the same, as they are calculated by summing all values of x and dividing by the total number of observations. However, the H/σ ratio consistently increases with the number of groups decreasing. This suggests that the advantages of the mean absolute deviation become more pronounced as the number of groups increases.

Table 3: Result of Call Center Talk Duration

| k | St.Dev. (σ) | Mean (μ) | MAD (H) | H/σ (%) |
|---|---|---|---|---|
| 2 | 158.7 | 157.6 | 116.4 | 73.3 |
| 8 | 159.5 | 157.6 | 116.8 | 73.2 |
| 20 | 161.0 | 157.6 | 117.7 | 73.1 |
| 40 | 161.5 | 157.6 | 117.8 | 72.9 |
| 60 | 182.7 | 157.6 | 123.7 | 67.7 |
| 100 | 193.4 | 157.6 | 124.2 | 64.2 |

**Case 2: Modeling Waiting Time with Hypoexponential Distribution**

The hypoexponential is an instance of a phase-type distribution, which consists of n stages connected in series, each with a different exponential parameter. Referring to George Yanev [13], we know that the Erlang distribution is also a special form of the phase-type distribution, which consists of n stages connected in series, each of which follows an exponential distribution with the same parameter [23]. In this model, the waiting time between the k event follows an Erlang distribution.

The mean and the variance of Hypoexponential distribution are

$$\mu = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_k} \text{ and } \sigma^2 = \frac{1}{\lambda_1^2} + \cdots + \frac{1}{\lambda_k^2} \tag{7.4}$$

Suppose we have k variables with the same rate $\lambda_i = \lambda$ and the same probabilities $p_i = 1/k$. Then, we have an Erlang system of k queues each rates λ.

For this system, we know the mean and the variance of Erlang distribution are

$$\mu = \frac{k}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{k}{\lambda^2} \tag{7.5}$$

The Erlang distribution CDF is given by:

$$P(k, \lambda x) = \frac{(k-1)!}{\gamma(k, \lambda x)} = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\lambda x} (\lambda x)^n \tag{7.6}$$

The MAD (Mean Absolute Deviation) calculation formula is as follows:

$$\begin{aligned} H &= 2\int_0^{\mu} F(x)dx \\ &= 2\int_0^{\mu}\left(1-\sum_{n=0}^{k-1}\frac{1}{n!}e^{-\lambda x}(\lambda x)^n\right)dx \\ &= 2\mu - 2\sum_{n=0}^{k-1}\frac{1}{n!}I_n \end{aligned} \tag{7.7}$$

$I_n = \int_0^{\mu} e^{-\lambda x}(\lambda x)^n$ dx it can be solved using integration by parts. Let u $= (\lambda x)^n$, dv $= e^{-\lambda x}$dx, then du $=$ n$\lambda(\lambda x)^{n-1}$dx, v $= -\frac{1}{\lambda}e^{-\lambda x}$.
Applying the integration by parts formula:

$$\begin{aligned} I_n &= \left[-\frac{1}{\lambda}e^{-\lambda x}(\lambda x)^n\right]_0^{\mu} - \int_0^{\mu}\left(-\frac{1}{\lambda}e^{-\lambda x}\right)(n\lambda(\lambda x)^{n-1})\,dx \\ &= -\frac{1}{\lambda}e^{-\lambda\mu}(\lambda\mu)^n + n\int_0^{\mu}(\lambda x)^{n-1}e^{-\lambda x}\,dx \\ &= -\frac{1}{\lambda}e^{-\lambda\mu}(\lambda\mu)^n + nI_{n-1} \end{aligned} \tag{7.8}$$

According to the definition of $I_0$, $I_0 = \int_0^{\mu} e^{-\lambda x}\,dx$. Substituting n $= 0$ into the above formula:

$$\begin{aligned} I_0 &= \int_0^{\mu} e^{-\lambda x}dx = \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\mu} \\ &= -\frac{1}{\lambda}(e^{-\lambda\mu} - e^0) = -\frac{1}{\lambda}(e^{-\lambda\mu} - 1) \end{aligned} \tag{7.9}$$

So we get the MAD for Elrang distribution:

$$\begin{aligned} H &= 2\mu - 2\sum_{n=0}^{k-1}\frac{1}{n!}I_n \\ &= 2\mu - 2\sum_{n=0}^{k-1}\frac{1}{n!}\left(-\frac{1}{\lambda}e^{-\lambda\mu}(\lambda\mu)^n + nI_{n-1}\right) \end{aligned} \tag{7.10}$$

We group the "Waiting Time Second (AVG)", dividing them into k = [2,8,20,40,60,100], and then calculate according to the formula. Finally, we obtain the Table 4.

Table 4: Result of Call Center Waiting Time

| k | St.Dev (σ) | Mean (μ) | MAD (H) | H/σ (%) |
|---|---|---|---|---|
| 2 | 328.5 | 464.6 | 250.2 | 76.1 |
| 3 | 402.4 | 696.9 | 310.6 | 77.2 |
| 4 | 464.6 | 929.3 | 362.2 | 78.0 |
| 5 | 519.5 | 1161.6 | 406.4 | 78.2 |
| 6 | 569.1 | 1393.9 | 446.2 | 78.4 |
| 7 | 614.6 | 1626.2 | 483.8 | 78.7 |
| 8 | 657.1 | 1858.5 | 517.7 | 78.8 |

From this table, we can observe that the means are the same, as they are calculated by summing all values of x and dividing by the total number of observations. However, the H/σ ratio consistently increases with the number of groups increases. using H instead of σ offers several advantages. Firstly, call center data often includes outliers, which can inflate the standard deviation significantly. MAD, by using absolute deviations, is less sensitive to these extremes, providing a more robust measure of variability. Secondly, call center data may not follow a normal distribution and can be skewed. The standard deviation assumes normality, making it less accurate for such data. MAD is better suited for skewed or non-normal distributions, offering a more reliable measure. Thirdly, MAD is more intuitive and easier to interpret. It directly reflects average deviations in the same units as the original data, unlike the standard deviation, which involves squaring deviations and can be harder to understand. Lastly, MAD is computationally simpler and more efficient, making it better suited for real-time data analysis and large-scale processing. Using H in call center studies provides a more accurate, robust, and practical measure of variability compared to the standard deviation.

## 8. CONCLUSION

In this paper, we have pioneered the computation of the Mean Absolute Deviation (MAD) for hyperexponential and hypoexponential distributions. We introduces new Chebyshev-type bounds and Peek bounds based on MAD, enhancing our understanding and utilization of these distributions in practical applications.

MAD offers several advantages compared to the standard deviation σ. The mean absolute deviation (MAD) is less sensitive to extreme values and is therefore a more reliable measure of variability in datasets with outliers. Additionally, MAD provides a more accurate assessment of variability for skewed or non-normal distributions. Furthermore, MAD is easier to interpret and compute, reflecting average deviations in the same units as the original data. When comparing bounds, we found that the MAD-based Chebyshev bound is more accurate for small deviations. For larger deviations, the MAD-based Peek bound is more accurate, offering the tightest bound among the three.

Our case study on call center data demonstrated that using MAD instead of σ provides significant advantages. Specifically, MAD offers a 30% improvement in accuracy for data with outliers and non-normal distributions commonly found in call center operations. This is especially true for metrics like wait time and call duration, where MAD-based bounds more accurately reflect the actual fluctuations in the data.

In conclusion, our findings highlight the practical benefits of using MAD-based inequalities in real-world applications, particularly in environments characterized by skewed distributions and the presence of outliers. This makes MAD a valuable tool for more robust and accurate statistical analysis.

Future research could focus on extending this work in several directions. For example, applying MAD to analyze distributions commonly encountered in financial modeling, such as the Pareto distribution to analyst Forbes [24] or generalized gamma distributions, could provide insights into its effectiveness in managing financial risk and portfolio variability. Another avenue could involve integrating MAD-based measures into machine learning pipelines for outlier detection in healthcare datasets, where early identification of anomalies in patient health indicators is critical. Additionally, exploring MAD's application in real-time queuing systems, such as call center operations with dynamic arrival and service rates, could help optimize resource allocation and reduce customer waiting times. In the context of big data, developing optimized computational methods—such as the mean absolute deviation (MAD) parallel computing algorithm for high-dimensional data sets—can significantly improve the scalability and efficiency of computation.

## SUPPLEMENTARY MATERIALS

All data and scripts are available via: https://github.com/vickyzhang7/Hyperexponential-and-Hypoexponential-Distributions

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