A COUNTEREXAMPLE TO A CONJECTURE ON THE K-FLOW PROBLEM

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ABSTRACT

Seymour [4] conjectures that if the k-flow problem with demands at the sink nodes has a solution then there is a ½-integer-valued solution. In this paper we present a counterexample obtained by combining two elements of the Graver test set.

1. INTRODUCTION

In [4] the k-flow problem is stated as follows: Suppose G = (V, E) is a graph, F $\subseteq$ E, each e$\in$ E–F has a capacity w(e) (integer-valued) and each f$\in$ F has a demand q(f) (integer-valued). When does there exist, for each f$\in$ F, a flow Θ_f in (V, E-F) between the ends of f and of value q(f), such that $\forall$ e$\in$ E-F, $\sum_{f\in F}\left|\Theta_f(e)\right| \leq w(e)$? The conjecture in [4] is: For any G, F, q and w, if the flows Θ_f (f $\in$ F) exist at all, then they can always be chosen ½-integer-valued. (F is the set of the sink-source edges, and is only necessary to ensure that Kirchoff's Law (refer [2]) also holds at the sinks and sources.)

Graver test set was developed to understand the nature of integer linear programming problems (refer [1]). These test sets allow repeated search for improved solutions through the interior of the feasible region. The structure of the test sets also sheds light on the nature of the solution (refer [3]).

We present a counterexample to this conjecture by constructing two elements of the Graver test set (refer [1]) for the 5-commodity flow problem. While proving our result, we present some of the structure of the k-flow problem by an analysis of the test sets.

2. GRAPH G AND THE 5-FLOW

Consider the graph G in Figure 1. Labels ±1, ±2, ±3 and ±4 are repeated twice indicating that the nodes represented by each of these labels must be identified as one node. The capacity of each of the displayed edges is 1. The sink-source edges -1.1, -2.2, -3.3, -4.4 and -5.5 have not been shown but the demand of each of the first 4 edges is 2 while the demand for -5.5 is 1. The commodities are represented by X, Y, U, V and Z respectively.

One solution is presented in Figure 1 itself, where X, Y, U and V take ¼-integer values on some edges. We will show that though the 5-flow commodity problem on G has a solution there is no ½-integer-valued solution.

3. Definitions, Notations and Observations

Note 1: The k-flow problem has 2 sets of constraints – Kirchoff's Law and the capacity constraints. An implicit constraint is that along any sink-source edge f $\in$ F, the flow of any commodity other than the corresponding commodity equals 0.

The capacity constraints may be replaced by $\forall$ e$\in$ E-F, $\sum_{f\in F}(\pm f(e)) \leq w(e)$.
For the 5-commodity flow problem, this is equivalent to:

$\forall$ e$\in$ E-F, $\pm X(e) \pm Y(e) \pm Z(e) \pm U(e) \pm V(e) \leq w(e)$.

[1] defines the **Support** of a k-flow as those expressions on the left-hand side of the capacity constraints that are non-zero. We can consider a subset of these expressions for the Support by dropping those expressions where the coefficient of any one commodity, say the first commodity, is -1, as these are redundant. For example, in the 2-flow problem, if X(e)+Y(e) is non-zero so is (–X(e)-Y(E)). Let S be the set of expressions in the left hand side of the capacity constraints excluding the expressions with -1 as the coefficient of the first commodity. Note that $|S| = 2^{|F|-1} * |E-F|$

Definitions: Adapting the definitions in [1] to the k-flow problem:

1. The **Support** of a k-flow Θ is defined as Supp$\Theta = \{s : s \in S, s \neq 0\}$.
2. A k-flow Θ is **Minimal** if SuppΘ is a minimal subset in {Supp $\Omega : \Omega \neq 0$, Ω is an integral k-flow} and Θ is not an integral multiple of any other integral k-flow. The set of minimal flows forms the test set.

For the 5-commodity flow, it is sufficient to examine $X(e) \pm Y(e) \pm Z(e) \pm U(e) \pm V(e)$.

Note 3: Consider a 3-flow Θ. To compute Supp Θ we need to analyze the four expressions $(X \pm Y \pm Z)(e)$ for every edge e.

For any edge e:

- If exactly 2 of these expressions equal 0, then one of X, Y or Z equals 0 on edge e. The remaining commodities have equal flow, and depending on the 2 expressions, these 2 commodities flow in the same direction or opposite directions on e.
- If 3 of these expressions equal 0, then the 4th is 0 and there is no flow on e.
- If all 4 expressions equal 0, there is no flow on e.
- If exactly one expression equals 0, then the sum of 2 of the commodities equals the third and depending on the expression, this sum (taken as a 1-commodity flow) and the 3rd commodity flow in the same direction or opposite directions on e.
- Nothing can be said if no expression equals 0.

Notation: In graph G, let V_1 be the set of nodes labeled 6 through 17 together with $\{\pm 1, \pm 2, \pm 5\}$ and let V_2 be the set of nodes labeled 18 through 29 with $\{\pm 1, \pm 2, \pm 5\}$. Let ϕ_1 be 4Φ restricted to the graph $G_1=(V_1, E_1)$ and ϕ_2 be 4Φ restricted to the graph $G_2=(V_2, E_2)$. Note that ϕ_1 and ϕ_2 are integral 3-flows and Φ = 1/4*($\phi_1+\phi_2$). We will show that ϕ_1 and ϕ_2 are minimal 3-flows.

Note 4: The following three sets are cuts in G_1.
C1 = {2.9, -1.8, 15.14, 14.13, 13.12} separating {1, -2} from {-1, 2}
C2 = {16.8, 16.17, 9.17, 1.15, -2.12} separating {1, -2} from {-1, 2}
C3 = {16.17, 2.13, 1.15, 9.10, 9.8} separating {1, 2} from {-1, -2}

Note 5: In [3] it has been shown that the only minimal 2-flows that do not increase the flows in the sink-source edges are 2-flows along elementary cycles.
For example, the elementary cycles in G_1 are 8.9.10.12.13.14.15.7.8, 8.16.17.9.10.12.13.14.15.7.8, 8.16.17.9.8. The minimal flows are 1 unit of X and Y respectively flowing in the same direction or opposite directions in each of these cycles.

4.The Counter Example

Theorem 1: $\Phi = 1/4*(\phi_1+\phi_2)$ and ϕ_1 and ϕ_2 are minimal 3-flows.

Proof: By construction it is clear that ϕ_1 and ϕ_2 are 3-flows and □$\Phi = 1/4*(\phi_1+\phi_2)$. We will show that ϕ_1 is a minimal flow. Since G and Φ is symmetric with respect to (X,Y) and (U,V), it will follow that ϕ_2 is also a minimal flow.

Suppose ϕ_1 is not minimal. By Theorem 3.12 of [1], $\phi_1 = \sum_{i=0}^{k} \alpha_i v_i$, where

1. The v_i are distinct minimal 3-flows
2. Each v_i is compatible with ϕ_1. That is, for each edge e and each v_i, $(X \pm Y \pm Z)(e)$ has the same sign as ϕ_1. In other words, on each edge e, $(X \pm Y \pm Z)(e)$ taken as a 1-flow flows in the same direction for all v_i and ϕ_1.
3. Supp $v_i \subset$ Supp ϕ_1
4. Each α_i is positive

Since Z = ½ on the edge 5.6 in ϕ_1, at least one v_i has Z flowing on 5.6. Let this minimal flow be denoted by θ.
By Kirchoff's Law and the construction of G_1, this Z flow in θ must return along 5.-5. That means there is a Z flow in 5.6.7 and 10.11.-5. On the other hand, because of Kirchoff's Law and the fact that X(-5.5) = Y(-5.5) = 0 neither X nor Y can flow in any of the edges in 5.6.7 or 10.11.-5. Therefore Supp θ = Supp ϕ_1 on 5.6.7 and 10.11.-5. Since θ is compatible with ϕ_1, Z flows from 5 to 7 and 10 to -5 in θ……..............................…**(1)**

Consider all the edges above or below the path 7.8.9.10. On these edges, either $X+Y \pm Z = 0$ or $X-Y \pm Z = 0$ for ϕ_1, and therefore for θ. This implies that for θ, Z = 0 in all these edges. Therefore, in θ, Z flows only in 5.6.7.8.9.10.11.-5 from 5 to -5. ………………………………..……………………………..…………………**(2)**

Consider the edge 8.16. Since $X+Y \pm Z = 0$ in ϕ_1, the same is true for θ. If either of $X-Y \pm Z = 0$ equals 0 in θ, then by Note 3, nothing will flow in this edge. Therefore for θ and this edge, either Suppθ = Suppϕ_1 or nothing flows. If there is 0-flow in 16.8, then X = 0 in 16.17 since X cannot flow in 16.-2.2. This implies Y = 0 in 16.17 otherwise Suppθ ⊃ Supp$\phi_1$on 16.17. This implies that Y = 0 in 9.17 which implies X = 0 in this edge. Using similar arguments for the other edges in path 8.16.17.9, we see that on 8.16.17.9, either Suppθ = Suppϕ_1 or $\theta = 0$.
Similarly for the path 10.12.13.14.15.7, Suppθ = Suppϕ_1 or $\theta = 0$.

If θ is 0 on 10.12.13.14.15.7, then X = Y = 0 in this path, and, since X and Y cannot flow in -5.5, X = Y = 0 in the edges 7.8 and 9.10. Since Z flows in these two edges, this implies that Suppθ ⊄ Suppϕ_1! Therefore θ is not 0 on 10.12.13.14.15.7 and Suppθ= Suppϕ_1 on 10.12.13.14.15.7**…………………………………………………………..(3)**

Suppose θ = 0 in 8.16.17.9. For θ, (X+Y-Z) = 0 on the edges 7.8 and 9.10 since this equation holds for ϕ_1. By (2), a non-zero Z flows from 7 to 8 and from 9 to 10. This implies that (X+Y) taken as a 1-flow, must flow from 7 to 8 and from 9 to 10. On the other hand, on edge 8.9, Z flows from 8 to 9 and (X+Y+Z) = 0. This implies that (X+Y) must flow from 9 to 8. If θ is 0 in 8.16.17.9 then Kirchoff's Law w.r.t (X+Y) is violated at nodes 8 and 9. Therefore Supp θ = Supp ϕ_1 on the path 8.16.17.9**……………..........(4)**

We now show Supp θ = Supp ϕ_1 on the path 7.8.9.10. By (2), (3) and the fact that θ is compatible with ϕ_1, equal volumes, say q, of X and Y flow from 15 to 7 and from 10 to 12 respectively. Since these flows cannot flow to 6 from 7 or from 11 to 10, these q units of X and Y flow from 7 to 8 and from 9 to 10. Since (X+Y-Z) = 0 on 7.8 and 9.10 in $\phi_1$and therefore in θ, by (2) Z = 2q from 7 to 10. Therefore on 7.8 and 9.10, Supp θ = Supp ϕ_1. On 8.9, (X+Y+Z) = 0 for ϕ_1 and θ, and 2qZ flows from 8 to 9. For Supp θ to be a proper subset of Supp ϕ_1 on 8.9, Note 3 requires exactly 2q units of either X or Y must flow from 9 to 8. Consider the vertex 9 and (X+Y) as a single flow. (X+Y) = 0 on 9.17 (since it is 0 in ϕ_1), 2q from 9 to 10 (since X = Y = q) and 2q from 9 to 8. Therefore there is a 4q (X+Y) flow from 2 to 9 which is a Y-flow since only Y flows on 2.9. Of this 4q units of Y, q units flow from 9 to 10. Suppose 2q units of Y flows from 9 to 8 (and therefore X = 0 on this edge). In which case, the remaining q units of Y flow along 9.17.16. Since Supp θ = Supp ϕ_1 on the path 8.16.17.9, q units of X also flows from 17 to 16 (originating from 1.17). This q units of X continues from 16 to 8. On the other hand, the 2 q units of Y from 9 to 8 combines with the q units of Y from 7 to 8 to form 3 q units of Y flowing from 8 to 16. This implies Supp θ ≠Supp ϕ_1 on 8.16, contradicting (4). Similarly X = 2q from 9 to 8 requires X = 3q from 17 to 9 which requires Y = 3q from 9 to 17 to 16 which requires X = 3q from 17 to 16 to 8. But from 8 to 16 there is only q units of Y flowing from 7 to 8 to 16 contradicting Supp θ = Supp ϕ_1 on the path 8.16.17.9. Therefore both X and Y flow from 8 to 9 and Supp θ = Supp ϕ_1 on the 8.9**……………………………………………………………………….…………..…..….(5)**

Consider the edge 1.15. We know that only X and Y flow in the path 7.15.14.13.12.10. Since Supp θ = Supp ϕ_1 on 7.15.14.13.12.10, taking (X+Y) as a 1-flow, (X+Y) = 0 on 14.15 and (X+Y) ≠ 0 on 15.7. This implies that there is flow in 1.15. Since only X can flow in 1.15, Supp θ = Supp ϕ_1 on 1.15. Similarly we can show that Supp θ = Supp ϕ_1 on all edges emanating from a source or a sink**…………………...……..…..…………...…..(6)**

Hence Supp θ = Supp ϕ_1 on G_1. This implies there is no integral 3-flow in G_1 with support properly contained in ϕ_1. Theorem 1 follows.

Theorem 2: No solution exists where all flows take integer or ½-integer values.
Proof: Suppose there exists another solution Φ* where all the flows are ½-valued.

In Φ*, Z cannot flow from 5 to -5 on any path different from that in Φ. Suppose not:

- If Z flows in 5.6.7.15.14.13.12.10.11.-5 with volume 1 or ½ (by assumption), the capacity of the cut C1 = {2.9, -1.8, 15.14, 14.13, 13.12} separating {1, -2} from {-1, 2} is 2 or 3.5. By the Max Flow Min Cut Theorem, Φ* is sub-optimal.

- If Z flows in 5.6.7.8.16.17.9.10.11.-5 with volume 1 or ½, the capacity of the cut C2 = {16.8, 16.17, 9.17, 1.15, -2.12} separating {1, -2} from {-1, 2} is 2 or 3.5. By the Max Flow Min Cut Theorem, Φ* is sub-optimal.

Similarly we can prove that in Φ* Z cannot flow in the paths 5.18.19.27.26.25.24.22.23.-5 and 5.18.19.20.28.29.21.22.23.-5.

In Φ*, ½ unit of Z flows in 5.6.7.8.9.10.11.-5 and 5.18.19.20.21.22.23.-5. Suppose not:

- If Z is greater than ½ in 5.6.7.8.9.10.11.-5, the capacity of the cut C3 = {16.17, 2.13, 1.15, 9.10, 9.8} separating {1, 2} from {-1, -2} is less than 4, and by Max Flow Min Cut Theorem, any XY-flow is < 4.

Similarly one can show that Z cannot be greater than ½ in 5.18.19.20.21.22.23.-5.

Since 1 unit of Z flows from 5 to -5, the Z flow in Φ* is identical to the Z flow in Φ.

In Φ*, 0 units of U/V flow in G_1 and 0 units of X/Y flow in G_2.
Suppose U flows in G_1. By assumption, U = 0, ½ or 1 in G_1. The proof for the paths 5.6.7.15.14.13.12.10.11.-5 and 5.6.7.8.16.17.9.10.11.-5 are identical to the case for the Z flow. For the path 5.6.7.8.9.10.11.-5, since ½ unit of Z is already flowing in this path, U must also equal ½ in these edges and the entire path 5.6.7.8.9.10.11.-5 is saturated. Then the cut C3 = {16.17, 2.13, 1.15, 9.10, 9.8} separating {1, 2} from {-1, -2} has capacity = 3. In which case Φ* is sub-optimal. We may similarly prove for the other commodities.

Therefore we have for Φ*:
1. ½ unit of Z flows in each of 5.6.7.8.9.10.11.-5 and 5.18.19.20.21.22.23.-5.
2. (X,Y) is restricted to G_1 and (U,V) is restricted to G_2.

We want to show that the flows of X,Y,U and V cannot be 0, ½ or 1 on every edge.

Consider the unit X flow in 1.15. Suppose:
1. *At vertex 15, 1 unit of X flows to 7*: This flow must continue from 7 to 8. Which is impossible since ½ unit of Z already flows in 7.8 and the capacity of 7.8 is 1.
2. *At vertex 15, 1 unit of X flows to 14:* Then the 1 unit of Y flow in 2.13 must flow to 12 to -2. Therefore both 14.15 and 12.13 are saturated. Then the unit X and Y flows in edges 1.17 and 2.9 respectively, must flow in the subgraph defined by {7, 8, 9, 10, 16, 17, $\pm 1, \pm 2$}. But the cut {8.9, 16.17} separates {1, 2} from {-1, -2} in this subgraph and has capacity 1.5 since ½ unit Z flows in 8.9 making Φ* is sub-optimal.
3. *At vertex 15, ½ unit of X flows to 7 (and the other to 14)*: This flow must continue through 7.8 where ½ Z also flows. This means that 7.8 is saturated. Then the 1 unit of Y flow in 2.13 must flow to 12 to -2. As above, this forces Φ* to be sub-optimal

Therefore in the edges of 7.14.15, X cannot be 0, ½ or 1. And hence the theorem.

Corollary 3: If Φ* is another solution, then Φ*-Φ is a sum of minimal circular 2-flows.
Proof: By Theorem 2 Φ*-Φ comprises two 2-flows – a (X,Y) restricted to G_1 and a (U,V) restricted to G_2 with no flow in the sink-source edges. By Theorem 3.1.2 [1], Φ*-Φ is the sum of minimal flows. Note 5 shows that these minimal flows are minimal circular 2-flows.

References

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Figure 1