A NEW STUDY OF TRAPEZOIDAL, SIMPSON'S1/3 AND SIMPSON'S 3/8 RULES OF NUMERICAL INTEGRAL PROBLEMS.

Md. Jashim Uddin[1], Mir Md. Moheuddin[2] and Md. Kowsher[1]

[1]Dept. of Applied Mathematics, Noakhali Science and Technology University, Noakhali-3814, Bangladesh.
[2]Dept. of CSE (Mathematics), Atish Dipankar University of Science and Technology (ADUST), Dhaka-1230, Bangladesh.

ABSTRACT

The main goal of this research is to give the complete conception about numerical integration including Newton-Cotes formulas and aimed at comparing the rate of performance or the rate of accuracy of Trapezoidal, Simpson's 1/3, and Simpson's 3/8. To verify the accuracy, we compare each rules demonstrating the smallest error values among them. The software package MATLAB R2013a is applied to determine the best method, as well as the results, are compared. It includes graphical comparisons mentioning these methods graphically. After all, it is then emphasized that the among methods considered, Simpson's 1/3 is more effective and accurate when the condition of the subdivision is only even for solving a definite integral.

KEYWORDS

Integration, Newton-Cotes formula, Trapezoidal method, Simpson's 1/3 method, Simpson's 3/8 method.

1. INTRODUCTION

Integration, which is a process of measuring the area plotted on a graph by a function as follows,

$$I = \int_a^b f(x)dx$$

is the total value or summation of f(x) dx over the range from a to b. The system of estimating the value of a definite integral from the approximate numerical values of the integrand, known as numerical integration. A function of a single variable which is exerted in numerical integration is called quadrature as well as expresses the area under the curve f(x). Besides, there are no singularities of the integrand in the domain under the assumption and also numerical integration comprises a boarding family of algorithms for the sake of counting the numerical values of a definite integral. Nowadays, it is essential due to computers are too able to go through the analytic manner of integration, even associating between analytical schemes and computer processor.

In 1915, the term 'Numerical Integration' had first demonstrated in the publication of A Course in Interpolation as well as Numeric Integration for the Mathematical Laboratory by David Gibb. There are several application fields in numerical integration as like applied mathematics, statistics, economics, and engineering, etc. Various methods are available in numerical integration, for example, Quadrature methods, Gaussian integration, Monte-Carlo integration, Adaptive Quadrature, and Euler-Maclaurin formula which are used to calculate those functions that are not integrated so easily. The various formula of numerical integrations is recounted in the books of S.S. Sastry[11], R.L. Burden[12], J.H. Mathews[13] as well as numerous other authors. J. Oliver [14] investigated the several processes of evaluation of definite integrals using higher-order formula. Besides, Gerry Sozio [15] discussed a detailed summary of different techniques of numerical integration. Using Bayesian methods, numerical integration is engaged in estimating likelihoods and posterior distributions [16].

Moreover, the value of definite integral $\int_a^b y\, dx$ which is enumerated by replacing the function y using an interpolation formula and then integrated between a and b. In such a way, we can obtain quadrature formula for which numerical values are acquainted as well. In many practical circumstances, it is inevitable and more necessary than numerical differentiation.

In our working procedure, we have investigated and also compared with the existing some Newton-Cotes methods such as the Trapezoidal rule, Simpson's 1/3 rule as well as Simpson's 3/8 rule to achieve the best results among them. Moreover, we demonstrated some sub-interval randomly for determining the integral numerically and applied numerical examples to compare our solutions with the exact value showing some condition graphically to obtain the effective method which gives lesser error value among the mentioned methods.

A. PURPOSE AND MOTIVES OF THE STUDY

The main purpose of this study is to evaluate the method which is the best for solving the definite function applying numerical methods. The objectives of the study are given below,

- Estimating the low error value of a solution, convergence as well as accurate results from the other remaining methods.
- Comparing the existing methods for computing the appropriate method concerning the given problems.

2. RELATED WORK

In the present era, numerical integration plays an extremely significant role in mathematics affiliate, still, it is one of the branches joining the analytical calculations as well as computer analysis. On the other hand, a large number of researchers have already been done comprehensive research tasks with a view to modeling and promoting the several fields of numerical integration for different objectives.

Besides, for instance, Ohta et al. [1] have compared various numerical integration to search out the most effective method for the Kramers-Kronig transformation, applying the analytical formula of the Kramers-Kronig transformation of a Lorentzian function as a reference. Also, they compared their methods including the application of (1) Maclaurin's formula, (2) trapezium formula, (3) Simpson's formula, and (4) successive double Fourier transform methods. In [2], Siushansian, R. et al. demonstrated how the convolution integral arising in the electromagnetic constitutive relation can be approximated by the trapezoidal rule of numerical integration as well as implemented using a newly derived one-time-step recursion relation. Moreover, in their paper, they have presented a comparison of different time-domain numerical techniques to model material dispersion. However, Pennestrì et al. [3] gave and compared eight widespread engineering friction force models, focused the attention on well-known friction models as well as delivered a review and comparison based on numerical efficiency.

What is more, in [4], Uilhoorn et al. attempted to search a fast and robust time integration solver to obtain gas flow transients within the framework of particle filtering and investigated both stiff and nonstiff solvers, namely embedded explicit Runge–Kutta (ERK) schemes. Bhonsale et al. [5] basically presented a comparison between three different numerical solution strategies for breakage population balance models and their results achieved for the fixed pivot technique, moving pivot technique and the cell average technique. Furthermore, these approaches, Concepcion Ausin, M. [6] compared various numerical integration producers and examined about more advanced numerical integration procedures. In [7], Rajesh Kumar Sinha et al. have worked to estimate an integrable polynomial discarding Taylor Series.

To solve Optimal Control Problem, Docquier, Q. et al. [8] explored the different dynamic formulations and compared their performances and their focus had on minimal coordinates and the derivation of the dynamics via the recursive methods for tree-like MBS (i.e., the so-called Newton-Euler and Order-N recursive algorithms). In their paper, they introduced different formulations and discussed their derivations. In [9], Parisi, V. et al. approach the classical, Newtonian, gravitational N-body problem utilizing a new, original numerical integration method and give the new algorithm, which is used to a set of sample cases of initial conditions in the `intermediate' N regime (N=100). Yet Brands, B. et al [10] have tested the comparison of the aforementioned hyper-reduction techniques focusing on accuracy and robustness, the well-known DEIM is disapproved for their application as it suffers from serious robustness deficiencies.

Unlike these works, we discussed and investigated the most general one, namely the Newton-Cotes methods involving the Trapezoidal, Simpson's 1/3 and Simpson's 3/8 rules. Several procedures compared and endeavoured to display better methods with a few error values among the existing methods, even to estimate the more proper values of definite integrals.

3. MATERIALS AND METHODS

Because of these tasks, the following methods had compared; Trapezoidal method, Simpson's 1/3 method, and Simpson's 3/8 method.

A. GENERAL QUADRATURE FORMULA

Let I = $\int_a^b y\,dx$, where y=f(x).Let f(x) be given for certain equidistant values of x=x_0, x_0+2h ,...., x_0+kh. Suppose y_0,y_1,……,y_k are the entries corresponding to the arguments x_0= a, x_1= a+h, x_2= a+2h ,....., x_k= a+kh = b respectively. Then we obtain,

$$\therefore I = \int_a^b y dx = \int_{x_0}^{x_0+kh} y_x\, dx$$

We know, u = $\frac{x-x_0}{h}$

or x= x_0 + uh ∴ dx = hdu

Limits: When x=x_0, then u=0

When x= x_0+kh, then u = k

$$\therefore I = \int_{x_0}^{x_0+kh} y\,dx = \int_0^k y_{x_0+uh}\, h\,du$$

$$= h\int_0^k \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)(u-n+1)}{n!}\Delta^k y_0 \right] du$$

$$= h[ky_0 + \frac{k^2}{2}\Delta y_0 + (\frac{k^3}{3} - \frac{k^2}{2})\frac{\Delta^2 y_0}{2!} + (\frac{k^4}{4} - k^3 + k^2)\ \frac{\Delta^3 y_0}{3!} + (\frac{k^5}{5} - \frac{3k^4}{2} + \frac{11k^3}{3} - 3k^2)\frac{\Delta^4 y_0}{4!} + \cdots]$$

This is the required Newton-Cotes method i.e, general quadrature formula. When k = 1, 2, 3...... then we obtain the Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule respectively. There are some graphical examples of Newton-Cotes where the integrating function can be polynomials for any order-for instance, (a) straight lines or (b) parabolas. The integral can be approximated in one step or in a series of steps to develop accuracy as,

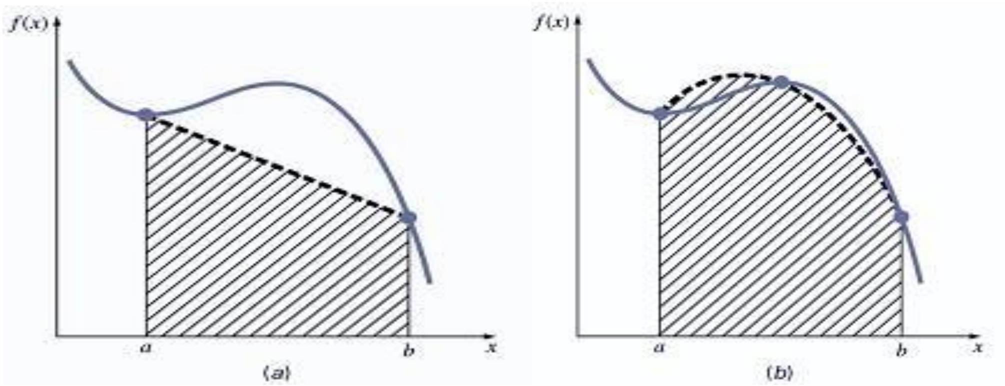

Figure. 1 Graphical examples of Newton-Cotes.

B. The General Formula of Trapezoidal Rule

In numerical analysis, the trapezoidal rule or method is a idea for approximating the definite integral, the average of the left and right sums as well as usually imparts a better approximation than either does individuallyThe basic idea of Trapezoidal rule graph is below .

$$I = \int_{x_0}^{x_k} f(x)dx$$

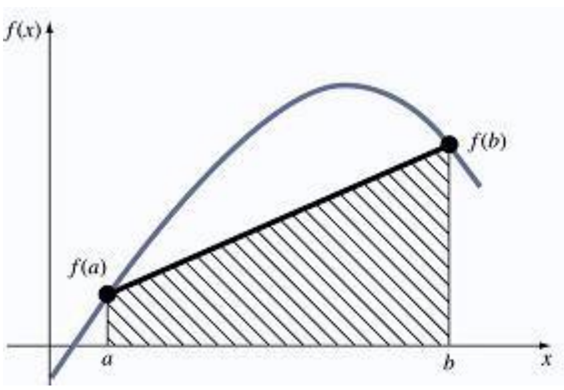

Figure. 2 Trapezoidal rule.

Also, we know from Newton-Cotes general quadrature formula that

$$I = h\left[ky_0 + \frac{k^2}{2}\Delta y_0 + \left(\frac{k^3}{3} - \frac{k^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{k^4}{4} - k^3 + k^2\right)\frac{\Delta^3 y_0}{3!} + \left(\frac{k^5}{5} - \frac{3k^4}{2} + \frac{11k^3}{3} - 3k^2\right)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

Now, putting k =1 in the above formula and neglecting the second and higher difference we get,

$$\int_{x_0}^{x_0+h} y\ dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{1}{2}h\left[(y_0 + y_1)\right]$$

Similarly, $\int_{x_{0+h}}^{x_0+2h} y\ dx = \frac{1}{2}h\,(y_1 + y_2)$

…………………………
…………………………

$$\int_{x_0+(k-1)h}^{x_0+kh} y\ dx = \frac{1}{2}h\,(y_{k-1} + y_k)$$

Adding these all integrals, we get,

$$\int_{x_0}^{x_0+kh} y\ dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \dots\dots + y_{k-1}) + y_k\right]$$

This rule is acquainted as the trapezoidal rule.

C. The General Formula of Simpson's One-Third Rule

In numerical integration, the Simpson's 1/3 rule is a numerical scheme for discovering the integral $\int_a^b y\ dx$ within some finite limits a and b. Simpson's 1/3 rule approximates f(x) with a polynomial of degree two p(x), i.e a parabola between the two limits a and b, and then searches the integral of that bounded parabola which is applied to exhibit the approximate integral $\int_a^b y\ dx$. Besides, Simpson's one-third rule is a tract of trapezoidal rule therein the integrand is approximated through a second-order polynomial. The basic idea of Simpson's one-three graph is as follows:

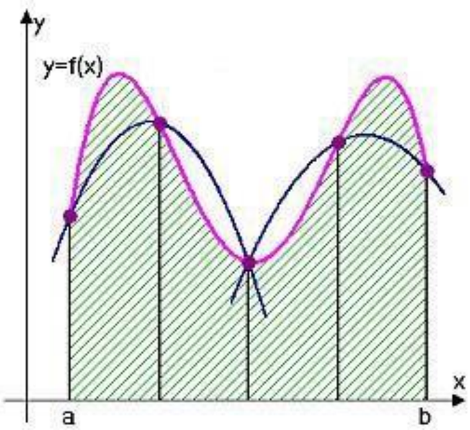

Figure. 3 Simpson's 1/3 rule.

Now, we know from Newton-Cotes general quadrature formula that

$$I = h\left[k y_0 + \frac{k^2}{2}\Delta y_0 + \left(\frac{k^3}{3} - \frac{k^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{k^4}{4} - k^3 + k^2\right)\frac{\Delta^3 y_0}{3!} + \left(\frac{k^5}{5} - \frac{3k^4}{2} + \frac{11k^3}{3} - 3k^2\right)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

Putting k =2 in the formula and neglecting the third and higher difference we get,

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{\left(\frac{8}{3}-2\right)}{2}\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{1}{3}h\,(y_0 + 4y_1 + y_2)$$

Similarly, $\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{1}{3}h(y_2 + 4y_3 + y_4)$

……………………………
……………………………

$$\int_{x_0+(k-2)h}^{x_0+kh} y\,dx = \frac{1}{3}h(y_{k-2} + 4y_{k-1} + y_k)$$

When k is even.
Adding these all integrals, we obtain,

$$\int_{x_0}^{x_0+2h} y\,dx + \int_{x_0+2h}^{x_0+4h} y\,dx + \cdots + \int_{x_0+(k-2)h}^{x_0+kh} y\,dx$$

$$= \frac{1}{3}h\left[(y_0 + y_k) + 4(y_1 + y_3 + \cdots + y_{k-1}) + 2(y_2 + y_4 + \cdots + y_{k-2})\right]$$

Or, $\int_{x_0}^{x_0+kh} y\,dx = \frac{h}{3}[(y_0 + y_k) + 4(y_1 + y_3 + \ldots + y_{k-1}) + 2(y_2 + y_4 + \ldots + y_{k-2})]$.

This formula is known as Simpson's one-third rule. If the number of sub-divisions of the interval is even then this method is only applied.

D. THE GENERAL FORMULA OF SIMPSON'S THIRD-EIGHT RULE

Simpson's three-eight rule is a process for approximating a definite integral by evaluating the integrand at finitely many points and based upon a cubic interpolation rather than a quadratic interpolation. The different is Simpson's 3/8 method applies a third-degree polynomial(cubic) to calculate the curve. The basic idea of Simpson's three eighth's graph is as follows:

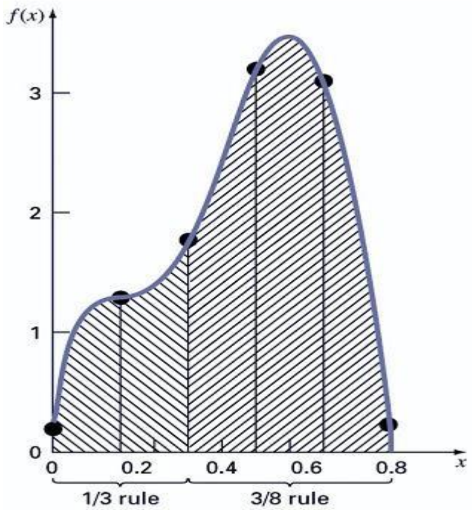

Figure. 4 Simpson's 3/8 rule.

Further, we know from Newton-Cotes general quadrature formula that

$$I = h\left[ky_0 + \frac{k^2}{2}\Delta y_0 + \left(\frac{k^3}{3} - \frac{k^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{k^4}{4} - k^3 + k^2\right)\frac{\Delta^3 y_0}{3!} + \left(\frac{k^5}{5} - \frac{3k^4}{2} + \frac{11k^3}{3} - 3k^2\right)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

Putting k =3 in the formula and neglecting all differences above the third, we get,

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{3!}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$\int_{x_0}^{x_0+3h} y\,dx = \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly, $\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3}{8}h(y_3 + 3y_4 + 3y_5 + y_6)$

……………………………
……………………………

$$\int_{x_0+(k-3)h}^{x_0+kh} y\,dx = \frac{3}{8}h(y_{k-3} + 3y_{k-2} + 3y_{k-1} + y_k)$$

Adding these all integrals, we get,

$$\int_{x_0}^{x_0+3h} y\,dx + \int_{x_0+3h}^{x_0+6h} y\,dx + \cdots + \int_{x_0+(k-3)h}^{x_0+kh} y\,dx$$

$$= \frac{3}{8}h\left[(y_o + y_k) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{k-1}) + 2(y_3 + y_6 + \ldots + y_{k-3})\right]$$

This formula is known as simpson’s three-eights rule.

4. RESULTS AND DISCUSSION

Problem-1: Suppose $\int_0^{\pi/2} \sin(x)\, dx$that is determined by using Simpson's 1/3 rule, Simpson's3/8 rule & Trapezoidal rule and interpreting the results by the three methods in this tasks, the results of the methods are demonstrated in the following table as well as the comparison of the approximate error is also given below.

K	Exact value	Simpson's 1/3 rule	Error	Simpson's 3/8 rule	Error	Trapezoidal rule	Error
1	1.000000	0.5235987	0.47640	0.589048	0.41095	0.785398	0.21460
2	**1.000000**	**1.0022798**	**0.00228798**	0.919304	0.08070	0.948059	0.05194
3	1.000000	0.8258986	0.17410	1.0010049	0.0010049	0.977048	0.02295
4	**1.000000**	**1.0001345**	**0.0001345**	0.900821	0.09918	0.987115	0.01288
5	1.000000	0.8953350	0.10466	0.961517	0.03848	0.991761	0.00824
6	**1.000000**	**1.0000263**	**0.0000263**	1.0000596	0.00006	1.251620	0.25162
7	1.000000	0.9252143	0.07479	0.943693	0.05631	0.995800	0.00420
8	**1.000000**	**1.0000082**	**0.0000082**	0.975634	0.02437	0.996785	0.0032
9	1.000000	0.9418275	0.05817	1.0000116	0.0000116	0.997460	0.00254
10	**1.000000**	**1.0000033**	**0.0000033**	0.960656	0.03934	0.997942	0.00206
11	1.000000	1.1428019	0.14280	0.982216	0.01778	1.140373	0.14037
12	**1.000000**	**1.0000016**	**0.0000016**	1.0000016	0.0000016	0.998571	0.00143
13	1.000000	1.1208316	0.12083	0.969758	0.03024	1.119173	0.11917
14	**1.000000**	**1.00000088**	**0.00000088**	0.986006	0.01399	0.998950	0.00105
15	1.000000	1.1047204	0.10472	1.0000015	0.0000015	1.103518	0.10352
16	**1.000000**	**1.00000051**	**0.00000051**	0.975437	0.02456	0.999196	0.00080
17	1.000000	1.0924001	0.09240	0.988467	0.01153	1.091491	0.09149
18	**1.000000**	**1.00000032**	**0.00000032**	1.00000032	0.00000032	1.086465	0.08647
19	1.000000	0.9724424	0.02756	0.979320	0.02068	0.999430	0.00057
20	**1.000000**	**1.00000021**	**0.00000021**	0.990193	0.00981	0.999485	0.00051
21	1.000000	0.9750668	0.02493	1.00000039	0.00000039	0.999533	0.00047
22	**1.000000**	**1.00000014**	**0.00000014**	0.982142	0.01786	1.070884	0.07088
23	1.000000	1.0682956	0.06830	0.991469	0.00853	1.067827	0.06783
24	**1.000000**	**1.00000010**	**0.00000010**	1.00000022	0.00000022	1.065022	0.06502
25	1.000000	0.9790561	0.02094	0.984287	0.01571	0.999670	0.00033
26	**1.000000**	**1.000000074**	**0.000000074**	0.992452	0.00755	1.060055	0.06006
27	1.000000	0.9806075	0.01939	1.00000014	0.00000014	0.999717	0.00028
28	**1.000000**	**1.000000055**	**0.000000055**	0.985971	0.01403	0.999737	0.00026
29	1.000000	0.9819449	0.01806	0.993232	0.00677	0.999755	0.00024
30	**1.000000**	**1.000000041**	**0.000000041**	1.000000094	0.000000094	0.9997715	0.00023
31	1.000000	0.9831097	0.01689	0.987329	0.01267	0.999786	0.00021
32	**1.000000**	**1.000000032**	**0.000000032**	0.993866	0.00613	0.999799	0.00020
33	1.000000	0.9841333	0.01587	1.000000064	0.000000064	0.999811	0.00019
34	**1.000000**	**1.000000025**	**0.000000025**	0.988448	0.01155	0.999822	0.00018
35	1.000000	0.9850400	0.01496	0.994391	0.00561	0.999832	0.00017
36	**1.000000**	**1.000000020**	**0.000000020**	1.000000020	0.000000020	1.043453	0.04345
37	1.000000	0.9858486	0.01415	0.989384	0.01062	0.999849	0.00015
38	**1.000000**	**1.000000016**	**0.000000016**	0.994834	0.00517	1.041176	0.04118
39	1.000000	0.9865744	0.01343	1.000000032	0.000000032	0.999864	0.00014
40	**1.000000**	**1.000000013**	**0.000000013**	0.990181	0.00982	0.999871	0.00013

Table 1: The results of the three methods.

Here K = 1 to 40 which is the number of subdivision of the interval of the integration.

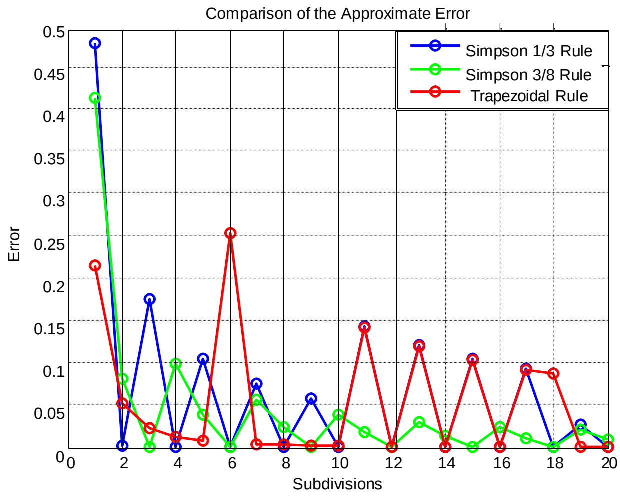

Figure. 5 The approximate error is plotted against the number of subdivisions (1-20).

From the above table and comparison of the approximate error, we can claim that Simpson's 1/3 rule gives the lesser error value among other methods when the condition of the subdivision is only even, other methods impart less accuracy in this case as compared to other methods. As a result, it is recommended strongly that Simpson's 1/3 is the most robust method for solving a definite integral and very close to the exact value.

Similarly,
Problem-2: Let $\int_0^6 e^x \quad dx$that is calculated by applying Simpson's 1/3 rule, Simpson's 3/8rule & Trapezoidal rule and interpreting the results by the three methods in this tasks, the results of the methods are demonstrated in the following table as well as the comparison of the approximate error is also given below.

K	Exact value	Simpson's 1/3 rule	Error	Simpson's 3/8 rule	Error	Trapezoidal rule	Error
1	402.428	808.85758	406.42958	909.96478	507.536	1213.28638	810.8576
2	**402.428**	**484.77094**	**82.34294**	522.77107	120.3423	666.89980	264.471
3	402.428	362.12087	76.30713	442.79280	40.36402	528.40320	125.9744
4	**402.428**	**411.29757**	**8.86957**	370.21766	32.2111	475.19813	72.76934
5	402.428	331.66775	70.76025	398.33387	4.09492	449.59961	47.17082
6	**402.428**	**404.42370**	**1.9957**	406.48342	4.054635	435.41858	32.98979
7	402.428	334.8533	67.5747	355.88397	46.5448	1007.08899	604.6602
8	**402.428**	**403.09146**	**0.66346**	399.33387	3.09492	421.11813	18.68934
9	402.428	342.02113	60.4077	403.32695	0.898158	417.22431	14.79552
10	**402.428**	**402.70657**	**0.27857**	361.32310	41.1057	755.9862	353.5582
11	402.428	348.79228	53.6365	388.37344	14.0554	712.22136	309.7926
12	**402.428**	**402.56447**	**0.13647**	402.72548	0.29669	410.77800	8.34921
13	402.428	354.53572	47.8931	367.25854	35.1703	409.54726	7.118476
14	**402.428**	**402.50259**	**0.07459**	389.3363	13.0925	408.56964	6.140854
15	402.428	359.31136	43.1174	402.55283	0.124044	407.78025	5.351466
16	**402.428**	**402.47227**	**0.04427**	372.06195	30.3668	407.13373	4.70495
17	402.428	363.29057	39.1382	390.50270	11.9261	579.11662	176.6878
18	**402.428**	**402.45603**	**0.02803**	402.48929	0.060501	567.22460	164.7958
19	402.428	366.63459	35.7942	375.83332	26.5955	405.76753	3.338742

20	**402.428**	**402.44671**	**0.01871**	391.58997	10.8388	547.64259	145.2138
21	402.428	369.47354	32.9552	402.46167	0.032882	539.48788	137.0591
22	**402.428**	**402.44105**	**0.01305**	378.82213	23.6067	532.19502	129.7662
23	402.428	371.90827	30.5205	392.54317	9.88562	404.70841	2.279624
24	**402.428**	**402.43746**	**0.00946**	402.44815	0.019362	404.52259	2.093807
25	402.428	374.01628	28.4125	381.23121	21.1976	404.35859	1.92981
26	**402.428**	**402.43509**	**0.00709**	393.36556	9.06323	404.21313	1.784347
27	402.428	375.85740	26.5714	402.44091	0.012125	404.08351	1.654728
28	**402.428**	**402.43348**	**0.00548**	383.20684	19.222	403.96752	1.538732
29	402.428	377.47817	24.9506	394.07416	8.35463	403.86330	1.434517
30	**402.428**	**402.43235**	**0.00435**	402.43676	0.007973	403.76932	1.340539
31	402.428	378.91522	23.5136	384.852820	17.576	490.10452	87.67573
32	**402.428**	**402.43154**	**0.00354**	394.68724	7.74155	403.60709	1.178304
33	402.428	380.19764	22.2312	402.43424	0.005454	484.20041	81.77162
34	**402.428**	**402.43095**	**0.00295**	386.24348	16.1853	403.47261	1.043827
35	402.428	381.34879	21.08	395.22094	7.20785	403.41385	0.985061
36	**402.428**	**402.43051**	**0.00251**	402.43264	0.003856	403.35991	0.931121
37	402.428	382.38764	20.0412	387.43295	14.9958	403.31028	0.881492
38	**402.428**	**402.43017**	**0.00217**	395.68863	6.74016	403.26451	0.835727
39	402.428	383.32968	19.042	402.43159	0.002802	403.22222	0.793436
40	**402.428**	**402.42992**	**0.00192**	388.46136	13.9674	403.18306	0.75506

Table 2: The results of the three methods.

Here K=1 to 40 which is the number of subdivision of the interval of the integration.

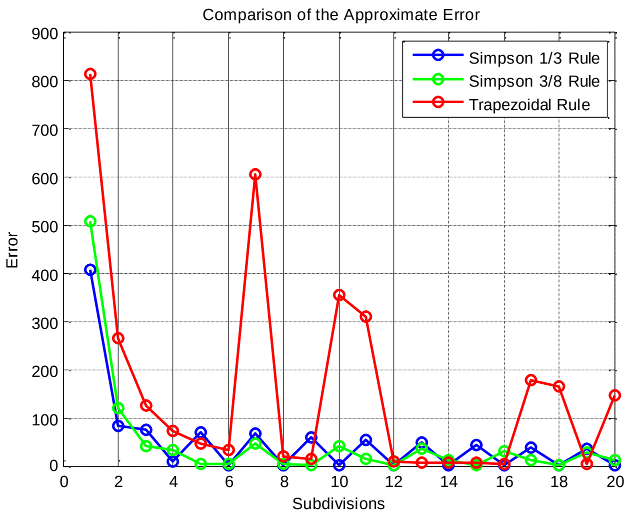

Figure. 6 The approximate error is plotted against the number of subdivisions (1-20).

From the above table and comparison of the approximate error, we can claim that Simpson's 1/3 rule gives the lesser error value among other methods when the condition of the subdivision is only even, other methods impart less accuracy in this case as compared to other methods. As a result, it is recommended strongly that Simpson's 1/3 is the most robust method for solving a definite integral and very close to the exact value.

A. VERIFICATION TO ACHIEVE THE BEST METHOD FOR PROBLEM-1 & PROBLEM-2 GRAPHICALLY

Firstly, we get the following graphical comparison for problem-1 when subinterval is 4.

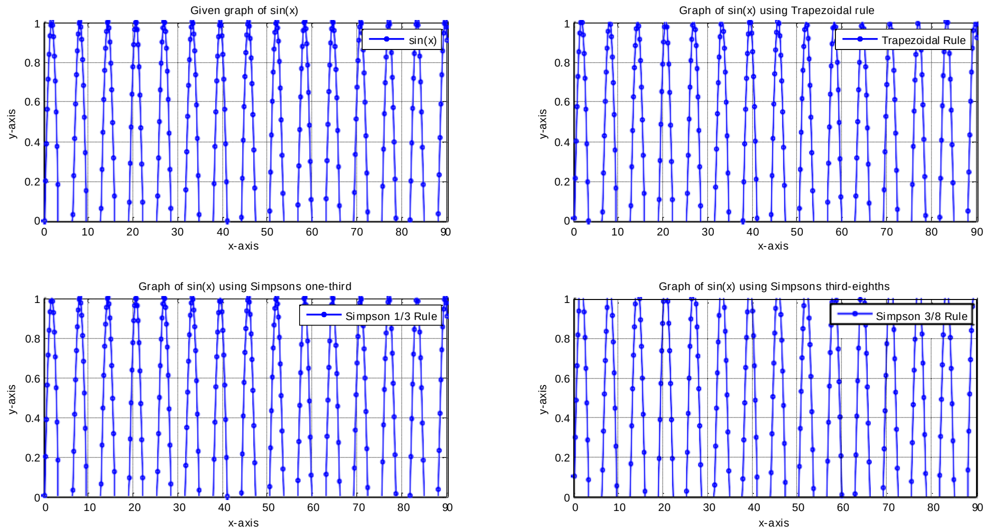

Figure.7 Graphical comparisons.

Comparing the above graph,we see that Simpson's 1/3 is a better method than others.
Similarly,
Secondly, we obtain the following graphical comparison for problem-2 when subinterval is 6.

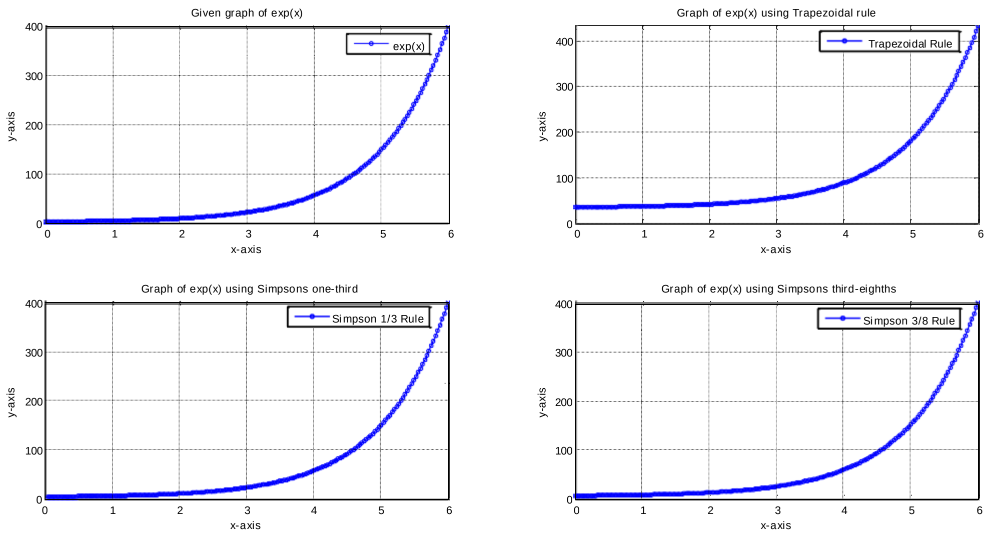

Figure. 8 Graphical comparisons.

Comparing the above graph,we see that Simpson's 1/3 is a better method than others.

5. CONCLUSION

From the methods examined in our paper, we are capable of showing numerical integration for finding the smallest error value by using the methods of Trapezoidal as well as Simpson's 1/3, Simpson's 3/8 rules that we have discussed. In our paper tasks, we have tried to display some examples as well as emphasized the condition for which Simpson's one-third method is the best. Consequently, we see that Simpson's one-third rule gives the smallest error value among the rules as well as formally it is the most effective and appropriate methods among the mentioned rules in the case of even subdivision. s

ACKNOWLEDGEMENT

We are especially thankful to reveal our heartiest gratitude and sincerest liability to Md. Jashim Uddin, Assistant Professor, Dept. of Applied Mathematics, Noakhali Science and Technology University for imparting us the valuable suggestion and constant encouragement to work on this research field.

REFERENCES

[1] Ohta, Koji, and Hatsuo Ishida. "Comparison among several numerical integration methods for Kramers-Kronig transformation." *Applied Spectroscopy* 42.6 (1988): 952-957.

[2] Siushansian, R., & LoVetri, J. (1995). A comparison of numerical techniques for modeling electromagnetic dispersive media. *IEEE Microwave and Guided Wave Letters*, *5*(12), 426-428.

[3] Pennestrì, Ettore, Valerio Rossi, Pietro Salvini, and Pier Paolo Valentini. "Review and comparison of dry friction force models." *Nonlinear dynamics* 83, no. 4 (2016): 1785-1801.

[4] Uilhoorn, Ferdinand Evert. "A comparison of numerical integration schemes for particle filter-based estimation of gas flow dynamics." *Physica Scripta* 93, no. 12 (2018): 125001.

[5] Bhonsale, S. S., Telen, D., Stokbroekx, B., & Van Impe, J. (2019). Comparison of numerical solution strategies for population balance model of continuous cone mill. *Powder Technology*.

[6] Concepcion Ausin, M. (2007) an introduction to quadrature and other numerical integration techniques, Encyclopedia of Statistics in Quality as well as reliability. Chichester, England.

[7] Rajesh Kumar Sinha, Rakesh Kumar, 2010, Numerical method for evaluating the integrable function on a finite interval, International Journal of Engineering Science and Technoligy. Vlo-2(6).

[8] Docquier, Q., Brüls, O., & Fisette, P. (2019). Comparison and Analysis of Multibody Dynamics Formalisms for Solving Optimal Control Problem. In *IUTAM Symposium on Intelligent Multibody Systems–Dynamics, Control, Simulation* (pp. 55-77). Springer, Cham.

[9] Parisi, V., & Capuzzo-Dolcetta, R. (2019). A New Method to Integrate Newtonian N-Body Dynamics. *arXiv preprint arXiv:1901.02856*.

[10] Brands, B., Davydov, D., Mergheim, J., & Steinmann, P. (2019). Reduced-Order Modelling and Homogenisation in Magneto-Mechanics: A Numerical Comparison of Established Hyper-Reduction Methods. *Mathematical and Computational Applications*, *24*(1), 20.

[11] S.S Sastry, 2007, Introductory Method of Numerical Analysis, Fourth Edition, Prentice-hall of India Private Limited.

[12] Richard L. Burden, 2007, Numerical Analysis, Seven Edition, International Thomson Publishing Company.

[13] Jonh H. Mathew, 2000, Numerical Method for Mathematics, science and Engineering, Second Edition, Prentice Hall of India Private Limited.

[14] J. Oliver, 1971, The evaluation of definite integrals using high-order formulae, The Computer Journal, Vol-14(3).

[15] Gerry Sozio, 2009, Numerical Integration, Australian Senior Mathematics Journal, Vol-23(1).

[16] Evans M, Swartz T. Methods for approximating integrals in statistics with special emphasis on Bayesian integration problems. Statistical Science. 19995;10:254-272.