ON β-NORMAL SPACES

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ABSTRACT

The aim of this paper is to study the class of β-normal spaces. The relationships among s-normal spaces, p-normal spaces and β-normal spaces are investigated. Moreover, we study the forms of generalized β-closed functions. We obtain characterizations of β-normal spaces, properties of the forms of generalized β-closed functions and preservation theorems.

1. INTRODUCTION

First step in normality was taken by Viglino [32] who de_ned semi normal spaces. Then Singal and Arya [28] introduced the class of almost normal spaces and proved that a space is 02010 Mathematics Subject Classi_cation: Primary : 54D10, Secondary : 54D15, 54A05, 54C08.

Key words and phrases. p-normal space, s-normal space, β-normal space, gβ-closed function, β-gβclosed function, separation axioms. normal if and only if it is both a semi-normal space and an almost normal space. Normality is an important topological property and hence it is of signi_cance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In recent years, many authors have studied several forms of normality [10, 12, 14, 24]. On the other hand, the notions of p-normal spaces and s-normal spaces were introduced by Paul and Bhattacharyya [27]; and Maheshwari and Prasad [17], respectively.

Levine [16] initiated the investigation of g-closed sets in topological spaces, since then many modi_cations of g-closed sets were de_ned and investigated by a large number of topologists [5, 7, 10, 25]. In 1996, Maki et al [19] introduced the concepts of gp-closed sets and Arya and Nour [4] introduced the concepts of gs-closed sets. The purpose of this paper is to study the class of

normal spaces, namely β-normal spaces, which is a generalization of the classes of p-normal spaces and s-normal spaces. The relations among β-normal spaces, p-normal spaces and s-normal spaces and also properties of β-normal spaces are investigated. Moreover, we study the forms of generalized _-closed functions. We obtain properties of these forms of generalized β-closed functions and preservation theorems.
Spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$ denotes a function f of a space (X, τ) into a space (Y, σ)). Let A be a subset of a space *X*. The closure and the interior of A are denoted by *cl*(A) and *int*(A) respectively.

De_nition 2.1. A subset A of a space X is called

(1) *regular open [29] if A = int(cl(A));*
(2) β-*open [22] if A* $\subseteq$ *int(cl(int(A)));*
(3) *semi-open [15] if A* $\subseteq$ *cl(int(A));*
(4) *β-open [1] if A* $\subseteq$ *cl(int(cl(A)));*
(5) *preopen [21] or nearly open [11] if A* $\subseteq$ *int(cl(A)).*

It is shown in [22] that the class of _-open sets is a topology and it is stronger than given topology on X.

The complement of an α-open (resp. semi-open, preopen, β-open, regular open) set is called α-closed [20] (resp. semi-closed [9], preclosed [21], β-closed [1], regular closed [29]).

The intersection of all α-closed (resp. semi-closed, preclosed, β-closed) sets containing A is called the α-closure (resp. semi-closure, preclosure, α-closure) of A and is denoted by αcl(A) (resp. *s*-cl(A), *p*-cl(A), β-cl(A)).

Dually, the α-interior (resp. semi-interior, preinterior, β-interior) of A, denoted by β-int(A) (resp. sint(A), pint(A), β-int(A)), is defined to be the union of all α-open (resp. semi-open, preopen, β-open) sets contained in A.

The family of all β-open (resp. β-closed, α-open, regular open, regular closed, semi-open, preopen) sets of a space X is denoted by βO(X) (resp. βC(X), βO(X), RO(X), RC(X), SO(X), PO(X)). The family of all β-open sets of X containing a point x is denoted by βO(X, x).

Lemma 2.2. [2] Let A be a subset of a space X and x 2 X. The following properties hold for β-cl(A):

(1) *x* € β-*cl*(A) if and only if A∩U 6= _ for every U € βO(X) containing x;
(2) A is β-closed if and only if A = β-cl(A);
(3) β-*cl*(A) $\subseteq$ β-cl(B) if A $\subseteq$ B;

(4) β-*cl*(_-cl(A)) = β-*cl*(A);
(5) β-*cl*(A) is β-closed.

De_nition 2.3. A space *X* is said to be prenormal [26] or p-normal [27] (resp. s-normal [17]) if for any pair of disjoint closed sets A and B, there exist disjoint preopen (resp. semi-open) sets U and V such that A $\subseteq$ U and B $\subseteq$ V.

De_nition 2.4. A subset A of a space $(X, \tau)$is said to be g-closed [16] (resp. gs-closed [4], gp-closed [19]) if cl(A) $\subseteq$ U (resp. s-cl(A) $\subseteq$ U, p-cl(A) $\subseteq$ U) whenever A_U and U € τ . The complement of g-closed (resp. gs-closed, gp-closed) set is said to be g-open (resp. gs-open, gp-open).

Definition 2.5. A subset A of a space (X, τ) is said to be sg-closed [5] (resp. pg-closed [6]) if s-cl(A) $\subseteq$ U (resp. p-cl(A) $\subseteq$ U) whenever A $\subseteq$ U and U € SO(X) (resp. U € PO(X)).

The complement of sg-closed (resp. pg-closed) set is said to be sg-open (resp. pg- pen).

3. β-NORMAL SPACES

Definition 3.1. [18] A space X is said to be β-normal if for any pair of disjoint closed sets A and B, there exist disjoint β-open sets U and V such that A $\subseteq$ U and B $\subseteq$ V.

Remark 3.2. The following diagram holds for a topological space (X, τ).

$$\begin{array}{ccc} \textit{normal} & \longrightarrow & \textit{s-normal} \\ \downarrow & & \downarrow \\ \textit{p-normal} & \longrightarrow & \beta\textit{-normal} \end{array}$$

None of these implications is reversible as shown by the following Examples.

Example 3.3. (1) *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\phi, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. *Then the space* (X, τ) *is* β*-normal but not p-normal.*

(2) *Let* $X = \{a, b, c, d, e\}$ *and* $\tau = \{\phi, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{b, c, d, e\}, X\}$. *Then the space* (X, τ) *is* β*-normal but not*

s-normal.

For the other implications the Examples can be seen in [11].

Theorem 3.4. For a space *X* the following are equivalent :

(1) X is β-normal,
(2) For every pair of open sets U and V whose union is X, there exist _-closed sets A and B such that A $\subseteq$ U, B $\subseteq$ V and A *U* B= X,
(3) For every closed set H and every open set K containing H, there exists a β-open setU such that H $\subseteq$ U $\subseteq$ β-cl(U) $\subseteq$ K.

Proof. (1) $\Rightarrow$ (2) : Let U and V be a pair of open sets in a β-normal space X such that $X = U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint closed sets. Since X is β-normal there exist disjoint β-open sets U_1 and V_1 such that $X \setminus U \subseteq U_1$ and $X \setminus V \subseteq V_1$. Let $A = X \setminus U_1$ and $B = X \setminus V_1$. Then A and B are β-closed sets such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(2) $\Rightarrow$ (3) : Let H be a closed set and K be an open set containing H. Then $X \setminus H$ and K are open sets whose union is X. Then by (2), there exist β-closed sets M_1 and M_2 such that $M_1 \subseteq X \setminus H$ and $M_2 \subseteq K$ and $M_1 \cup M_2 = X$. Then $H \subseteq X \setminus M_1$, $X \setminus K \subseteq X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = \phi$.

Let $U = X \setminus M_1$ and $V = X \setminus M_2$. Then U and V are disjoint β-open sets such that $H \subseteq U \subseteq X \setminus V \subseteq K$. As $X \setminus V$ is β-closed set, we have β-$cl(U) \subseteq X \setminus V$ and $H \subseteq U \subseteq \beta$-$cl(U) \subseteq K$.

(3) $\Rightarrow$ (1) : Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X \setminus H_2$, then $H_2 \cap K = \phi$. $H_1 \subseteq K$ where K is an open set. Then by (3), there exists a β-open set U of X such that $H_1 \subseteq U \subseteq \beta$-$cl(U) \subseteq K$. It follows that $H_2 \subseteq X \setminus \beta$-$cl(U) = V$, say, then V is β-open and $U \cap V = \phi$. Hence H_1 and H_2 are separated by β-open sets U and V. Therefore X is β-normal.

4. THE RELATED FUNCTIONS WITH β-NORMAL SPACES

Definition 4.1. A function f : X → Y is called

(1) pre β-open if f(U) 2 βO(Y) for each U €βO(X) [18];

(2) pre β-closed if f(U) 2 βC(Y) for each U €βC(X) [18];
(3) almost β-irresolute if for each x in X and each β-neighbourhood V of f(x), β-cl(f^{-1}(V)) is a β-neighbourhood of x.

Theorem 4.2. A function f : X → Y is pre β-closed if and only if for each subset A in Y and for each _-open set U in X containing f^{-1}(A), there exists a β-open set V of Y containing A such that f^{-1} (V) $\subseteq$ U.

Proof. (⇒): Suppose that f is pre β-closed. Let A be a subset of Y and $U \in \beta O(X)$ containing $f^{-1}(A)$. Put $V = Y \backslash f(X \backslash U)$, then V is a β-open set of Y such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.

(⇐) : Let K be any β-closed set of X. Then $f^{-1}(Y \backslash f(K)) \subseteq X \backslash K$ and $X \backslash K \in \beta O(X)$. There exists a β-open set V of Y such that $Y \backslash f(K) \subseteq V$ and $f^{-1}(V) \subseteq X \backslash K$. Therefore, we have $f(K) \supseteq Y \backslash V$ and $K \subseteq f^{-1}(Y \backslash V)$. Hence, we obtain $f(K) = Y \backslash V$ and $f(K)$ is β-closed in Y. This shows that f is pre β-closed.

Lemma 4.3. *For a function $f : X \to Y$, the following are equivalent:*

(1) *f is almost β-irresolute.*

(2) *$f^{-1}(V) \subseteq \beta\text{-}int(\beta\text{-}cl(f^{-1}(V)))$ for every $V \in \beta O(Y)$.*

Theorem 4.4. *A function $f : X \to Y$ is almost β-irresolute if and only if $f(\beta\text{-}cl(U)) \subseteq \beta\text{-}cl(f(U))$ for every $U \in \beta O(X)$.*

Proof. (⇒): Let $U \in \beta O(X)$. Suppose $y \notin \beta\text{-}cl(f(U))$. Then there exists $V \in \beta O(Y, y)$ such that $V \cap f(U) = \phi$. Hence, $f^{-1}(V) \cap U = \phi$. Since $U \in \beta O(X)$, we have $\beta\text{-}int(\beta\text{-}cl(f^{-1}(V))) \cap \beta\text{-}cl(U) = \phi$. Then by Lemma 4.3, $f^{-1}(V) \cap \beta\text{-}cl(U) = \phi$ and hence $V \cap f(\beta\text{-}cl(U)) = \phi$. This implies that $y \notin f(\beta\text{-}cl(U))$.

(⇐) : If $V \in \beta O(Y)$, then $M = X \backslash \beta\text{-}cl(f^{-1}(V)) \in \beta O(X)$. By hypothesis, $f(\beta\text{-}cl(M)) \subseteq \beta\text{-}cl(f(M))$ and hence $X \backslash \beta\text{-}int(\beta\text{-}cl(f^{-1}(V))) = \beta\text{-}cl(M) \subseteq f^{-1}(\beta\text{-}cl(f(M))) \subseteq f^{-1}(\beta\text{-}cl(f(X \backslash f^{-1}(V)))) \subseteq f^{-1}(\beta\text{-}cl(Y \backslash V)) = f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$. Therefore, $f^{-1}(V) \subseteq \beta\text{-}int(\beta\text{-}cl(f^{-1}(V)))$. By Lemma 4.3, f is almost β-irresolute.

Theorem 4.5. *If a surjective function $f : X \to Y$ is a pre β-open continuous almost β-irresolute from a β-normal space X onto a space Y, then Y is β-normal.*

Proof. Let A be a closed subset of Y and B be an open set of Y containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subseteq f^{-1}(B)$. As X is β-normal, there exists a β-open set U in X such that $f^{-1}(A) \subseteq U \subseteq \beta\text{-}cl(U) \subseteq f^{-1}(B)$ by Theorem 3.4. Then, $f(f^{-1}(A)) \subseteq f(U) \subseteq f(\beta\text{-}cl(U)) \subseteq f(f^{-1}(B))$. Since f is pre β-open almost β-irresolute surjection, we obtain $A \subseteq f(U) \subseteq \beta\text{-}cl(f(U)) \subseteq B$. Then again by Theorem 3.4 the space Y is β-normal.

Theorem 4.6. *If $f : X \to Y$ is a pre β-closed continuous function from a β-normal space X onto a space Y, then Y is β-normal.*|

Proof. Let M_1 and M_2 be disjoint closed sets of Y. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets of X. Since X is β-normal, then there exist disjoint β-open sets U and V such that $f^{-1}(M_1) \subseteq U$ and $f^{-1}(M_2) \subseteq V$. By Theorem 4.2, there exist β-open sets A and B such that $M_1 \subseteq A$, $M_1 \subseteq B$, $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Also A and B are disjoint. Thus, Y is β-normal.

Definition 4.7. *A function $f : X \to Y$ is called α-closed* [23] *if for each closed set in X, f(U) is α-closed set in Y.*

Theorem 4.8. If f : X → Y is an β-closed continuous surjection and X is normal, then Y is β-normal.

Proof. Let A and B be disjoint closed sets of Y . Then f^{-1} (A) and f^{-1} (B) are disjoint closed sets of X by the continuity of f. As X is normal, there exist disjoint open sets U and V in X such that f^{-1}(A) $\subseteq$ U and f^{-1} (B) $\subseteq$ V . By Proposition 6 in [23], there are disjoint _-open sets G and H in Y such that A $\subseteq$ G and B $\subseteq$ H. Since every _-open set is $\subseteq$ -open, G and H are disjoint β-open sets containing A and B, respectively. Therefore, Y is β-normal.

5. GENERALIZED B-CLOSED FUNCTIONS

Definition 5.1. [31] A subset A of a space (X, τ) is said to be gβ-closed if β-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U € τ.

De_nition 5.2. A subset A of a space (X, τ) is said to be _g-closed if β-cl(A) _ U whenever A $\subseteq$ U and U € βO(X).

The complement of βg-closed set is said to be βg-open.

Remark 5.3. The following diagram holds for any subset of a topological space X.

$$
\begin{array}{ccccc}
closed & & \longrightarrow & & g\text{-}closed \\
\downarrow & & & & \downarrow \\
semi\text{-}closed & \longrightarrow & sg\text{-}closed & \longrightarrow & gs\text{-}closed \\
\downarrow & & & & \downarrow \\
\beta\text{-}closed & \longrightarrow & \beta g\text{-}closed & \longrightarrow & g\beta\text{-}closed \\
\uparrow & & & & \uparrow \\
preclosed & \longrightarrow & pg\text{-}closed & \longrightarrow & gp\text{-}closed \\
\uparrow & & & & \uparrow \\
closed & & \longrightarrow & & g\text{-}closed
\end{array}
$$

None of these implications is reversible as shown by the following Examples and the related papers.

Example 5.4. *Let* $X = \{a, b, c, d, e\}$ *and* $\tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. *Then the set* $\{a, b\}$ *is* $g\beta$*-closed but not gp-closed.*

Example 5.5. *Let* X *and* τ *be as in Example 5.4. Then the set* $\{a, b\}$ *is* $g\beta$*-closed but not gs-closed.*

Example 5.6. *Let* X *and* τ *be as in Example 5.4. Then the set* $\{a, b, c, e\}$ *is* $g\beta$*-closed but it is not* βg*-closed.*

For the other implications the examples can be seen in [4, 5, 6, 9, 19, 21].

Definition 5.7. A function $f : X \rightarrow Y$ is said to be

(1) β-closed if f(A) is β-closed in Y for each closed set A of X [1],
(2) βg-closed if f(A) is βg-closed in Y for each closed set A of X,
(3) gβ-closed if f(A) is gβ-closed in Y for each closed set A of X.

Definition 5.8. A function $f : X \rightarrow Y$ is said to be

(1) quasi β-closed if f(A) is closed in Y for each A € βC(X),
(2) β-βg-closed if f(A) is βg-closed in Y for each A € βC(X),
(3) β-gβ-closed if f(A) is gβ-closed in Y for each A € βC(X) [31],
(4) almost gβ-closed if f(A) is gβ-closed in Y for each A € RC(X).

Remark 5.9. The following diagram holds for a function f : $(X, \tau) \rightarrow (Y, \sigma)$:

$$\begin{array}{ccccc}
\beta\text{-}g\beta\text{-}closed & \longrightarrow & g\beta\text{-}closed & \longrightarrow & almost\ g\beta\text{-}closed \\
\uparrow & & \uparrow & & \\
\beta\text{-}\beta g\text{-}closed & \longrightarrow & \beta g\text{-}closed & & \\
\downarrow & & \uparrow & & \\
\beta\text{-}closed & \longleftarrow & pre\ \beta\text{-}closed & & \\
\uparrow & & \uparrow & & \\
closed & \longleftarrow & quasi\ \beta\text{-}closed & &
\end{array}$$

Example 5.10. (1) *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{b, d\}, \{a, b, d\}, X\}$ *and* $\sigma = \{\phi, \{b\}, \{b, d\}, \{a, b, c\}, Y\}$. *Let* $f : (X, \tau) \rightarrow (Y, \sigma)$ *be an identity function. Then f is β-closed, βg-closed, gβ-closed but it is not β-gβ-closed, β-βg-closed, pre β-closed, closed.*

(2) *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, X\}$ *and* $\sigma = \{\phi, \{b\}, \{b, d\}, \{a, b, c\}, Y\}$. *Let* $f : (X, \tau) \rightarrow (Y, \sigma)$ *be an identity function. Then f is pre β-closed and closed but it is not quasi β-closed.*

Example 5.11. (1) *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a, c, d\}, X\}$ *and* $\sigma = \{\phi, \{b\}, \{b, d\}, \{a, b, c\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an identity function. Then f is almost* $g\beta$*-closed but it is not* $g\beta$*-closed.*

(2) *Let* $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, X\}$ *and* $\sigma = \{\phi, \{b\}, \{b, d\}, \{a, b, c\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an identity function. Then f is* $g\beta$*-closed but it is not* βg*-closed.*

(3) *Let* $X = \{a, b, c, d\}$ *and* $Y = \{a, b, c, d, e\}$. *Let* $\tau = \{\phi, \{b, d\}, \{a, b, d\}, X\}$ *and* $\sigma = \{\phi, \{e\}, \{a, b\}, \{a, b, e\}, \{c, d, e\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function defined as follows:* $f(a) = a$, $f(b) = b$, $f(c) = c$ *and* $f(d) = \{d, e\}$. *Then f is* β-$g\beta$*-closed but it is not* β-βg*-closed.*

Definition 5.12. A function f : X → Y is said to be β-gβ-continuous [30] if $f^{-1}(K)$ is gβ-closed in X for every K € βC(Y)

Definition 5.12. *A function* $f : X \to Y$ *is said to be* β-$g\beta$*-continuous* [30] *if* $f^{-1}(K)$ *is* $g\beta$*-closed in X for every* $K \in \beta C(Y)$.

Definition 5.13. *A function* $f : X \to Y$ *is said to be* β*-irresolute* [18] *if* $f^{-1}(V) \in \beta O(X)$ *for every* $V \in \beta O(Y)$.

Theorem 5.14. [31] *If* $f : X \to Y$ *is continuous* β-$g\beta$*-closed, then f(H) is* $g\beta$*-closed in Y for each* $g\beta$*-closed set H of X.*

Theorem 5.15. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be functions. Then*

(1) *the composition* $g \circ f : X \to Z$ *is* β-$g\beta$*-closed if f is* β-$g\beta$*-closed and g is continuous* β-$g\beta$*-closed.*

(2) *the composition* $g \circ f : X \to Z$ *is* β-$g\beta$*-closed if f is pre* β*-closed and g is* β-$g\beta$*-closed.*

Theorem 5.16. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be functions and let the composition* $g \circ f : X \to Z$ *be* β-$g\beta$*-closed. If f is a* β*-irresolute surjection, then g is* β-$g\beta$*-closed.*

Proof. Let $K \in \beta C(Y)$. Since f is β-irresolute and surjection, $f^{-1}(K) \in \beta C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Hence $g(K)$ is $g\beta$-closed in Z and hence g is β-$g\beta$-closed.

Lemma 5.17. [31] *A surjective function $f : X \to Y$ is $g\beta$-closed (resp. β-$g\beta$-closed) if and only if for each subset B of Y and each $U \in \tau$ (resp. $U \in \beta O(X)$) containing $f^{-1}(B)$, there exists a $g\beta$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Remark 5.18. *Every β-irresolute function is β-$g\beta$-continuous but not conversely.*

Example 5.19. *Let $X = \{a, b, c, d\}$ and $Y = \{a, b, c\}$. Let $\tau = \{\phi, \{c\}, \{b, c, d\}, X\}$ and $\sigma = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = a$, $f(b) = b$, $f(c) = f(d) = c$. Then f is β-$g\beta$-continuous function but it is not β-irresolute.*

Theorem 5.20. *A function $f : X \to Y$ is β-$g\beta$-continuous if and only if $f^{-1}(V)$ is $g\beta$-open in X for every $V \in \beta O(Y)$.*

Theorem 5.21. *If $f : X \to Y$ is closed β-$g\beta$-continuous bijection, then $f^{-1}(K)$ is $g\beta$-closed in X for each $g\beta$-closed set K of Y.*

Proof. Let K be a $g\beta$-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y, $K \subseteq V$, and $f^{-1}(V) \subseteq U$. Therefore, we have β-$cl(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(\beta\text{-}cl(K)) \subseteq f^{-1}(V) \subseteq U$. Since f is β-$g\beta$-continuous, $f^{-1}(\beta\text{-}cl(K))$ is $g\beta$-closed in X and hence $\beta\text{-}cl(f^{-1}(K)) \subseteq \beta\text{-}cl(f^{-1}(\beta\text{-}cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $g\beta$-closed in X.

Corollary 5.22. [31] *If $f : X \to Y$ is closed β-irresolute bijection, then $f^{-1}(K)$ is $g\beta$-closed in X for each $g\beta$-closed set K of Y.*

Theorem 5.23. *If $f : X \to Y$ is an open β-$g\beta$-continuous bijection, then $f^{-1}(K)$ is $g\beta$-closed in X for each $g\beta$-closed set K of Y.*

Proof. Let K be a $g\beta$-closed set of Y and U an open set of X containing $f^{-1}(K)$. Since f is an open surjective, $K = f(f^{-1}(K)) \subseteq f(U)$ and $f(U)$ is open in Y. Therefore, $\beta\text{-}cl(K) \subseteq f(U)$. Since f is injective, $f^{-1}(K) \subseteq f^{-1}(\beta\text{-}cl(K)) \subseteq f^{-1}(f(U)) = U$.

Since f is β-$g\beta$-continuous, $f^{-1}(\beta\text{-}cl(K))$ is $g\beta$-closed in X and hence $\beta\text{-}cl(f^{-1}(K)) \subseteq \beta\text{-}cl(f^{-1}(\beta\text{-}cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $g\beta$-closed in X.

Theorem 5.24. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition g o f : $X \to Z$ be β-$g\beta$-closed. If g is an open β-$g\beta$-continuous bijection, then f is β-$g\beta$-closed.*

Proof. Let $H \in \beta C(X)$. Then $(gof)(H)$ is $g\beta$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 5.23, $f(H)$ is $g\beta$-closed in Y and hence f is β-$g\beta$-closed.

Theorem 5.25. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition g o f : $X \to Z$ be β-$g\beta$-closed. If g is a closed β-$g\beta$-continuous injection, then f is β-$g\beta$-closed.*

Proof. Let $H \in \beta C(X)$. Then $(gof)(H)$ is $g\beta$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 5.21, $f(H)$ is $g\beta$-closed in Y and hence f is β-$g\beta$-closed.

6. Preservation theorems and other characterizations of β-normal spaces

Theorem 6.1. [31] *For a topological space X, the following are equivalent:*

(1) *X is β-normal,*

(2) *for any pair of disjoint closed sets A and B of X, there exist disjoint $g\beta$-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

(3) *for each closed set A and any open set B containing A, there exists a $g\beta$-open set U such that $A \subseteq U \subseteq \beta\text{-}cl(U) \subseteq B$,*

(4) *for each closed set A and any g-open set B containing A, there exists a β-open set U such that $A \subseteq U \subseteq \beta\text{-}cl(U) \subseteq int(B)$,*

(5) *for each closed set A and any g-open set B containing A, there exists a $g\beta$-open set G such that $A \subseteq G \subseteq \beta\text{-}cl(G) \subseteq int(B)$,*

(6) *for each g-closed set A and any open set B containing A, there exists a β-open set U such that $cl(A) \subseteq U \subseteq \beta\text{-}cl(U) \subseteq B$,*

(7) *for each g-closed set A and any open set B containing A, there exists a $g\beta$-open set G such that $cl(A) \subseteq G \subseteq \beta\text{-}cl(G) \subseteq B$.*

Theorem 6.2. *If $f : X \to Y$ is a continuous quasi β-closed surjection and X is β-normal, then Y is normal.*

Proof. Let M_1 and M_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is β-normal, there exist disjoint U_1, $U_2 \in \beta O(X)$ such that $f^{-1}(M_i) \subseteq U_i$ for $i = 1, 2$. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq U_i$ for $i = 1, 2$. Since $U_1 \cap U_2 = \phi$ and f is surjective we have $V_1 \cap V_2 = \phi$. This shows that Y is normal.

Lemma 6.3. [31] *A subset A of a space X is $g\beta$-open if and only if $F \subseteq \beta\text{-}int(A)$ whenever F is closed in X and $F \subseteq A$.*

Theorem 6.4. *Let $f : X \to Y$ be a closed β-$g\beta$-continuous injection. If Y is β-normal, then X is β-normal.*

Proof. Let N_1 and N_2 be disjoint closed sets of X. Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y. By the β-normality of Y, there exist disjoint V_1, $V_2 \in \beta O(Y)$ such that $f(N_i) \subseteq V_i$ for $i = 1, 2$. Since f is β-$g\beta$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $g\beta$-open sets of X and $N_i \subseteq f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \beta$-$int(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \beta O(X)$, $N_i \subseteq U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is β-normal.

Corollary 6.5. *If $f : X \to Y$ is a closed β-irresolute injection and Y is β-normal, then X is β-normal.*

Proof. This is an immediate consequence since every β-irresolute function is β-$g\beta$-continuous.

Lemma 6.6. *A function $f : X \to Y$ is almost $g\beta$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $g\beta$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Lemma 6.7. *If $f : X \to Y$ is almost $g\beta$-closed, then for each closed set M of Y and each $U \in RO(X)$ containing $f^{-1}(M)$, there exists $V \in \beta O(Y)$ such that $M \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Theorem 6.8. *Let $f : X \to Y$ be a continuous almost $g\beta$-closed surjection. If X is normal, then Y is β-normal.*

Proof. Let M_1 and M_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subseteq U_i$ for $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(M_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \phi$. By Lemma 6.7, there exists $V_i \in \beta O(Y)$ such that $M_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, where $i = 1, 2$. Since $G_1 \cap G_2 = \phi$ and f is surjective, we have $V_1 \cap V_2 = \phi$. This shows that Y is β-normal.

Corollary 6.9. *If $f : X \to Y$ is a continuous $g\beta$-closed surjection and X is normal, then Y is β-normal.*

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