Quantum Variation about Geodesics

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ABSTRACT

The caustic that occur in geodesics in space-times which are solutions to the gravitational field equations with the energy-momentum tensor satisfying the dominant energy condition can be circumvented if quantum variations are allowed. An action is developed such that the variation yields the field equations and the geodesic condition, and its quantization provides a method for determining the extent of the wave packet around the classical path.

KEYWORDS:

Geodesic congruence, caustics, quantum variations

MSC: 34C40, 49S05, 53C22

1. INTRODUCTION

The convergence of geodesic congruences in space-times satisfying the gravitational field equations with energy-momentum tensors satisfying the dominant energy condition is known to imply the existence of caustics such that the affine parameters on the curves do not have an infinite range. The problem of geodesic completeness of these space-times is a characteristic of all theories of gravity coupled to matter satisfying one of the energy conditions. While much work on quantum gravity has focussed on the absence of singularities in solutions to modified field equations, the effect of quantum variations on the space-times and paths in the manifolds remains to be determined. The existence of quantum variations about geodesics which may circumvent the problem of caustics is investigated, and the connection between the range of the affine parameter of the quantum trajectories on the type of fluctuation about the geodesic shall be elucidated.

The geodesic completeness of a space-time is established through the range of the affine parameter along a congruence. The geodesic equation may be derived from conservation equation for the energy-momentum tensor for worldlines of dust derived from invariance of the Lagrangian under reparameterization. A path integral with this action would define a quantum variation about the plane wave solutions representing free motion, and it follows from the replacement of the terms in the Raychaudhuri equation by expectation values in §2 that the deviation from the classical trajectory would be determined by the variance of the distribution. This technique is developed further in §3 and §4 for the gravitational theory by deriving the first-order differential system from a Lagrangian and then introducing quantization through the WKB approximation to give variations about the geodesics that would be sufficient to circumvent caustics.

The generalization of the Raychaudhuri equation through the introduction of a stochastic term is reviewed briefly in §5. The prediction of the range for a geodesic congruence is found to be related to the choice of a first-order formalism the expansion parameter and a second-order equation for the equilibrium probability distribution. Limits on the variations in stochastic quantization determine the viability of removing caustics. Another generalization to congruences on string worldsheets in §6 allows for an infinite range of the affine parameter for hyperbolic surfaces in the Euclidean formalism. The occurrence of these two-dimensional manifolds, including infinite-genus surfaces, in the quantization of string theory, provides a method, derived from a finite theory at microscopic scales, for the resolution of caustics in geodesic congruences and a restoration of the geodesic completeness of a space-time under phenomenologically realistic conditions.

2. Wave Packets and Geodesic Completeness

The fluctuations of the four-dimensional metric are considered likely to have an effect on the geodesic completeness theorems of classical relativity. While the space-time metric is derived from the Hilbert action, and geodesics extremize the arc-length integral, another derivation of geodesic equation follows from the conservation equation for the energyomentum tensor. Suppose that dust is added to the space-time $T_{\mu\nu}=\rho\ u_\mu\ u_\nu$. Conservation of the energy and momentum yields

$$\begin{aligned} u^\mu{}_{;\mu} &= 0 \\ u^\mu D_\mu u^\nu &= 0 \end{aligned} \qquad (2.1)$$

if u^μ is the tangent vector along a curve in a divergence-free timelike congruence. The existence of a continuous conservation law signifies a continuous symmetry of the Lagrangian.

The quantization of a Lagrangian describing the interaction of the gravitational field with particles of dust therefore may be used to define variations geodesic trajectories. The problem of the singularities and geodesic completeness of space-times that are solutions to the gravitational field equations can be investigated with the use of the Hamilton-Jacobi equation and the path integral.

In Feynman diagrams, ingoing and outgoing states are represented by straight lines in Minkowski space-time which are orthogonal to wave fronts defined by solutions to the field equations of definite momentum. In curved space-times, the ingoing and outgoing states would follow geodesics orthogonal to curved wavefronts representing solutions to the field equations. The Feynman rules can be derived from a path integral, and the functional integrals may be evaluated to give the scattering amplitudes. From the expansion in terms of generalized plane waves, the variation may be expressed by a sum over a range of indices representing the momenta and therefore the trajectories. This procedure, which is implemented typically in the interaction region, could be extended for the vacuous regions of space-time. Specifically, it should be possible to verify that the sum over momenta of the generalized plane waves, when substituted into the phase factor of the WKB approximation, satisfies the Hamilton-Jacobi equation.

The replacement of a plane wave of definite momentum by a wave packet still yields acceptable results. It remains to be determined whether this distribution about the geodesic is coincident with the computation of the expectation value ($u^\mu D_\mu u^\nu$) and the variance. If
$Y_{ab} = D_b u_a$,

$$\begin{aligned} u^c D_c Y_{ab} &= D_b(u^c D_c u_a) - (D_b u^c)(D_c u_a) + R_{cba}{}^d u^c u_d \\ &= -Y^c{}_b Y_{ac} + R_{cba}{}^d u^c u_d \end{aligned} \tag{2.2}$$

and, for classical geodesics [1],

$$\frac{d\theta}{ds} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}u^c u^d \tag{2.3}$$

where θ represents the expansion of the geodesic congruence. It must be proven that, if $u^c D_c u_a \neq \hat{0}$ for a trajectory, there exists a path in the wave packet such that $D^a(u^c D_c u_a) - R_{cd}\bar{u}^c\bar{u}^d > 0$ at some value of the affine parameter along the curve. If this inequality holds sufficiently often before caustics occur, the parameter can have an infinite range, and a measure of completeness with respect to quantum variations about geodesics is attained.

3. DERIVATION OF THE GEODESIC EQUATION FROM AN ACTION

Although the geodesic equation is a separate constraint from the field equations, both can be formulated in variables which render the system to be a coupled first-order set of differential equations. The gravitational and geodesic variables are

$$\vec{x} = \{\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha\}$$
$$\vec{u} = \{u_\alpha\} \tag{3.1}$$

where $\Sigma_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{H}$, $N_{\alpha\beta} = \frac{n_{\alpha\beta}}{H}$ and $A_\alpha = \frac{a_\alpha}{H}$, $\sigma_{\alpha\beta}$ is the shear, and $n_{\alpha\beta}$ and a_α represent the spatial curvature, and H is a scalar variable, while uα is the tangent vector field of the geodesics such that $u^\alpha u^\beta_{;\alpha} = 0$. The combined set of equations [2] is given by

$$\vec{x}' = f(\vec{x})$$
$$\vec{u}' = h(\vec{x}, \vec{u}). \tag{3.2}$$

Although $\vec{x}$ and $\vec{u}$ are defined by several variables, an action yielding the equations in variables corresponding to single vectors, may be constructed.

Theorem 3.1 An action exists such that the variation yields the gravitational field equations and the geodesic condition.

Proof.

Consider the action

$$\int d\tau \left[\vec{x}\cdot\vec{x}' - V_1(\vec{x},\vec{u}) + \vec{u}\cdot\vec{u}' - V_2(\vec{x},\vec{u})\right]. \tag{3.3}$$

$$\frac{d}{d\tau}(\vec{x}) + \frac{\partial V_1(\vec{x},\vec{u})}{\partial\vec{x}} + \frac{\partial V_2(\vec{x},\vec{u})}{\partial\vec{x}} - \vec{x}' = 0$$
$$\frac{d}{d\tau}(\vec{u}) + \frac{\partial V_1(\vec{x},\vec{u})}{\partial\vec{u}} + \frac{\partial V_2(\vec{x},\vec{x})}{\partial\vec{u}} - \vec{u}' = 0. \tag{3.4}$$

The cancellation of the first and last terms of these equations prevents the derivation of Eq.(3.2) from the variation of this action, whereas a quadratic function of $\vec{x}'$ or $\vec{u}'$ would produce a second derivative of the varibles in the equations.

A function $F(\vec{x},\vec{x}')$is required in the action such that

$$\frac{d}{d\tau}\left(\frac{\partial F(\vec{x},\vec{x}')}{\partial \vec{x}'}\right) - \frac{\partial F(\vec{x},\vec{x}')}{\partial \vec{x}} = \vec{x}'. \tag{3.5}$$

If

$$\frac{\partial F(\vec{x},\vec{x}')}{\partial \vec{x}} = k_1 \vec{x}\cdot\vec{x}' + F_1(\vec{x}'), \tag{3.6}$$

$$F(\vec{x},\vec{x}') = k\vec{x}\cdot\vec{x}' + F_1(\vec{x}'). \tag{3.7}$$

Then

$$\frac{\partial F(\vec{x},\vec{x}')}{\partial \vec{x}'} = k_1\vec{x}' + \frac{\partial F_1(\vec{x}')}{\partial \vec{x}'} \tag{3.8}$$

and

$$\frac{d}{d\tau}\left(\frac{\partial F(\vec{x},\vec{x}')}{\partial \vec{x}'}\right) = k_1\vec{x}' + \frac{d}{d\tau}\left(\frac{\partial F_1(\vec{x}')}{\partial \vec{x}'}\right). \tag{3.9}$$

It follows that

$$k_1\vec{x}' + \frac{d}{d\tau}\left(\frac{\partial F_1(\vec{x}')}{\partial \vec{x}'}\right) - k_1\vec{x}' = \vec{x}' \tag{3.10}$$

$$\frac{d}{d\tau}\left(\frac{\partial F_1(\vec{x}')}{\partial \vec{x}'}\right) = \vec{x}'. \tag{3.11}$$

If $F_1(\vec{x}')$ is proportional to $\vec{x}'$, the derivative vanishes. When $F_1(\vec{x}')$ is another function of $\vec{x}'$, the derivative is a function of $\vec{x}'$ and $\frac{d}{d\tau}$ introduces $\vec{x}''$.

When $F_1(\vec{x}') = \frac{1}{2}|\vec{x}'|^2$, the equations for $\vec{x}$ are

$$\vec{x}'' = -\frac{\partial V_1(\vec{x},\vec{u})}{\partial \vec{x}} - \frac{\partial V_2(\vec{x},\vec{u})}{\partial \vec{x}}. \qquad (3.12)$$

$$\vec{x}'' = \frac{df(\vec{x})}{d\vec{x}}\vec{x}' = f(\vec{x})\frac{df(\vec{x})}{d\vec{x}} = \frac{1}{2}\frac{d}{d\vec{x}}(f(\vec{x})^2) \qquad (3.13)$$

By Eq.(3.2),

$$\frac{\partial V_1(\vec{x},\vec{u})}{\partial \vec{x}} + \frac{\partial V_2(\vec{x},\vec{u})}{\partial \vec{x}} = -f(\vec{x})\frac{df(\vec{x})}{d\vec{x}}. \qquad (3.14)$$

And

$\tilde{F}(\vec{u},\vec{u}') = k_2\vec{u}\cdot\vec{u}' + \frac{1}{2}|\vec{u}'|^2$

Similarly, suppose that $\tilde{F}(\vec{u},\vec{u}') = k_2\vec{u}\cdot\vec{u}' + \frac{1}{2}|\vec{u}'|^2$ is included in the action. The equations in $\vec{u}$ would be

$$\vec{u}'' = -\frac{\partial V_1(\vec{x},\vec{u})}{\partial \vec{u}} - \frac{\partial V_2(\vec{u},\vec{u})}{\partial \vec{u}}. \qquad (3.15)$$

Suppose that

$$\begin{aligned} \vec{u}' &= h(\vec{x},\vec{u}) \\ \vec{u}'' &= \frac{\partial h(\vec{x},\vec{u})}{\partial \vec{x}}\vec{x}' + \frac{\partial h(\vec{x},\vec{x})}{\partial \vec{u}}\vec{u}' \\ &= \frac{\partial h(\vec{x},\vec{u})}{\partial \vec{x}}f(\vec{x}) + \frac{\partial h(\vec{x},\vec{u})}{\partial \vec{u}}h(\vec{x},\vec{u}). \end{aligned} \qquad (3.16)$$

The condition on the partial derivatives with respect to $\vec{u}$ is

$$\frac{\partial V_1(\vec{x},\vec{u})}{\partial \vec{u}} + \frac{\partial V_2(\vec{x},\vec{u})}{\partial \vec{u}} = -\frac{\partial h(\vec{x},\vec{u})}{\partial \vec{x}}f(\vec{x}) - \frac{\partial(\vec{x},\vec{u})}{\partial \vec{u}}h(\vec{x},\vec{u}). \qquad (3.17)$$

When the two potentials $V_1(\vec{x},\vec{u})$ and $V_2(\vec{x},\vec{u})$ satisfy the differential constraints (3.14) and (3.17), the system of equations (3.2) can derived from the variation of the integral

$$\int d\tau \left[k_1 \vec{x} \cdot \vec{x}' + \frac{1}{2}|\vec{x}'|^2 - V_1(\vec{x}, \vec{u}) + k_2 \vec{u} \cdot \vec{u}' + \frac{1}{2}|\vec{u}|^2 - V_2(\vec{x}, \vec{u}) \right]. \tag{3.18}$$

4. THE WKB APPROXIMATION TO RELATIVISTIC MOTION

The conjugate momenta are

$$\begin{aligned} \Pi_{\vec{x}} &= \frac{\partial L}{\partial \vec{x}'} = k_1 \vec{x} + \vec{x}' \\ \Pi_{\vec{u}} &= \frac{\partial L}{\partial \vec{u}'} = k_2 \vec{u} + \vec{u}'. \end{aligned} \tag{4.1}$$

and the Poisson brackts are

$$\begin{aligned} [\vec{x}(\tau), \Pi_{\vec{x}}(\tau')] &= k_1[\vec{x}(\tau), \vec{x}(\tau')] + [\vec{x}(\tau), \vec{x}(\tau')] = \delta(\tau - \tau') \\ [\vec{u}(\tau), \Pi_{\vec{u}}(\tau')] &= k_2[\vec{u}(\tau), \vec{u}(\tau')] + [\vec{u}(\tau), \vec{u}(\tau')] = \delta(\tau - \tau') \end{aligned} \tag{4.2}$$

The quantum commutators would be

$$\begin{aligned} [\vec{x}(\tau), \Pi_{\vec{x}}(\tau')] &= i\hbar\delta(\tau - \tau') \\ [\vec{u}(\tau), \Pi_{\vec{u}}(\tau')] &= i\hbar\delta(\tau - \tau'). \end{aligned} \tag{4.3}$$

The commutator of the identical field variables is non-zero on a timelike curve because the variables at different values of the affine parameter are causally related, while it vanishes at spacelike distances in higher dimensions.

Theorem 4.1 The quantum variations about the geodesics in the semiclassical limit can be determined by the phase cancellation in the wavefunction satisfying a time-independent Schr¨odinger equation.

Proof.

The Hamiltonian equals

$$\begin{aligned} H &= \Pi_{\vec{x}} \cdot \vec{x}' + \Pi_{\vec{u}} \cdot \vec{u}' - L \\ &= (k_1\vec{x} + \vec{x}') \cdot \vec{x}' + (k_2\vec{u} + \vec{u}') \cdot \vec{u}' - \left(k_1\vec{x} \cdot \vec{x}' + \frac{1}{2}\vec{x}' \cdot \vec{x}' - V_1(\vec{x}, \vec{u})\right) \\ &\quad - \left(k_2(\vec{u} \cdot \vec{u}' + \frac{1}{2}\vec{u} \cdot \vec{u}' - V_2(\vec{x}, \vec{u}\right) \\ &= \frac{1}{2}\vec{x}' \cdot \vec{x}' + \frac{1}{2}\vec{u}' \cdot \vec{u}' + V_1(\vec{x}, \vec{u}) + V_2(\vec{x}, \vec{u}) \\ &= \frac{1}{2}(\Pi_{\vec{x}} - k_1\vec{x})^2 + \frac{1}{2}(\Pi_{\vec{u}} - k_2\vec{x})^2 + V_1(\vec{x}, \vec{u}) + V_2(\vec{x}, \vec{u}). \end{aligned} \tag{4.4}$$

It consists only of the conjugate momenta terms and the potentials if $k_1 = 0 \text{ and } k_2 = 0.$

Substituting $P_{\vec{x}} \to i\hbar\frac{\partial}{\partial\vec{x}} \text{ and } P_{\vec{u}} \to i\hbar\frac{\partial}{\partial\vec{u}}$ yields the differential operator

$$-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\vec{x}^2} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial\vec{u}^2} + V_1(\vec{x}, \vec{u}) + V_2(\vec{x}, \vec{u}). \tag{4.5}$$

The time-independent Schodinger equation for particles of given energy is

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\vec{x}^2} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial\vec{u}^2} + V_1(\vec{x}, \vec{u}) + V_2(\vec{x}, \vec{u})\right]\psi = E\psi. \tag{4.6}$$

This differential equation has a form similar to that of the Wheeler-DeWitt equation for a wavefunction in a metric superspace. The Wheeler-DeWitt equation, $H\Psi = 0,$ derived from diffeomorphism constraints on the Lagrangian. It is consistent a Schrodinger equation with zero gravitational energy of a closed universe. However, the energy of a congruence would be nonvanishing, and it can be included.

From the WKB approximation $\psi(\vec{x}, \vec{u}) = A(\vec{x}, \vec{u})e^{i\frac{I(\vec{x}, \vec{u})}{\hbar}}$,

$$\begin{gathered} \frac{1}{2}\nabla_{\vec{x}}I \cdot \nabla_{\vec{x}}I + \frac{1}{2}\nabla_{\vec{u}}I \cdot \nabla_{\vec{u}}I + V_1(\vec{x}, \vec{u}) + V_2(\vec{x}, \vec{u}) = E + \frac{\hbar^2}{A}(\nabla_{\vec{x}} \cdot \nabla_{\vec{x}}A + \nabla_{\vec{u}} \cdot \nabla_{\vec{u}}A) \\ \nabla_{\vec{x}} \cdot (A^2\nabla_{\vec{x}}I) + \nabla_{\vec{u}} \cdot (A^2\nabla_{\vec{u}}I) = 0. \end{gathered} \tag{4.7}$$

Setting $I = I_0 + \hbar I_1 + \hbar^2 I_2 + ...$, it follows that

$$\begin{aligned}\frac{1}{2}\nabla_{\vec{x}} I_0 \cdot \nabla_{\vec{x}} I_0 + \frac{1}{2}\nabla_{\vec{u}} I_0 \cdot \nabla_{\vec{u}} I_0 + V_1(\vec{x},\vec{u}) + V_2(\vec{x},\vec{u}) = E \\ \nabla_{\vec{x}} I_0 \cdot \nabla_{\vec{x}} I_1 + \nabla_{\vec{u}} I_0 \cdot \nabla_{\vec{u}} I_1 = 0 \\ \nabla_{\vec{x}} I_0 \cdot \nabla_{\vec{x}} I_2 + \frac{1}{2}\nabla_{\vec{x}} I_1 \cdot \nabla_{\vec{x}} I_2 + \nabla_{\vec{u}} I_0 \cdot \nabla_{\vec{u}} I_2 + \frac{1}{2}\nabla_{\vec{u}} I_1 \cdot \nabla_{\vec{u}} I_1 = \frac{1}{2A}(\nabla_{\vec{x}} \cdot \nabla_{\vec{x}} A + \nabla_{\vec{u}} \cdot \nabla_{\vec{u}} A) \\ \vdots\end{aligned} \tag{4.8}$$

In the classical limit, $\hbar \to 0$, the first set of equations is

$$\nabla_{\vec{x}} I \cdot \nabla_{\vec{x}} I + \nabla_{\vec{u}} I \cdot \nabla_{\vec{u}} I + 2(V_1(\vec{x},\vec{u}) + V_2(\vec{x},\vec{u}) - E) = 0. \tag{4.9}$$

and the solutions are divided into the regions with $E > V_1(\vec{x},\vec{u}) + V_2(\vec{x},\vec{u})$ and $E < V_1(\vec{x},\vec{u}) + V_2(\vec{x},\vec{u})$ [3]. The wavefunctions are matched at the turning points $E = V_1(\vec{x},\vec{u}) + V_2(\vec{x},\vec{u})$.

The Hamilton-Jacobi equations yield the dependence of I on $\vec{x}$ and $\vec{u}$ about the classical configuration. The phases of ψ would add constructively the curve satisfying the geodesic equation in Eq.(3.2). The variation $\left.\frac{\partial I}{\partial u}\right|_{\vec{u}_{cl.}}$ does not vanish and there will be cancellation of the phases away from the geodesic. The average values of the position and velocity coordinates of a system of particles satisfy classical equations. The spread in the wave packet $\langle\vec{x}\rangle^2 - \langle\vec{x}\rangle^2$ and $(\Delta\vec{u})^2 = \langle\vec{u}\rangle^2 - \langle\vec{u}\rangle^2$ can be determined from the expectation values based on the wavefuntion $\psi = Ae^{i\frac{I}{\hbar}}$ and the solutions to Eq.(4.7).

It is known that the a system of free particles tends to disperse if $\langle H\rangle - E_{cl}$. is constant [3], while nonlinear terms in the potential would focus the congruence [4].

5. THE DEVELOPMENT OF CAUSTICS

It has been shown that the introduction of a Fokker-Planck equation for the three-metric yields the following solution for θ [5],

$$\theta(s) = \frac{3}{\left[\left(\frac{3}{|\theta(0)|}\right) - s + 3\int^s d\mathcal{B}(s)\right]}, \tag{5.1}$$

such that $|_{s\to\infty}\left[\frac{\mathcal{B}(s)}{2s\ log(log\ s))^{\frac{1}{2}}}\right]=1$. There is a divergence at approximately

$$\frac{3}{|\theta(0)|}-s+3(2slog(log\ s))^{\frac{1}{2}}=0 \qquad (5.2)$$

Implying

$$\left(\frac{3}{|\theta(0)|}-s\right)^2=9(2slog(log\ s)). \qquad (5.3)$$

The equation for the affine parameter is

$$s^2-\left[\frac{6}{|\theta(0)|}+18\ log(log\ s)\right]s+\frac{9}{|\theta(0)|^2}=0 \qquad (5.4)$$

which has the approximate solution

$$s=\left[\frac{3}{|\theta(0)|}+9log(log\ s)\right]\pm\left[\frac{3}{|\theta(0)|}+9log(log\ s)\right]\left\{1-\frac{\frac{9}{2|\theta(0)|^2}}{\left[\frac{3}{|\theta(0)|}+9\ log(log\ s)\right]^2}+...\right\} \ (5.5)$$

When $\frac{3}{|\theta(0)|}\ll 9log(log\ s)$, there is a singularity at

$$s|\approx\frac{\frac{3}{2|\theta(0)|^2}}{\left[\frac{1}{|\theta(0)|}+3log(log\ s)\right]}+...\ . \qquad (5.6)$$

Given the form of the solution (5.6) for s near a caustic at $s_0,\ |\theta(0)|\approx\frac{1}{\sqrt{2}\ s_0^{\frac{1}{2}}\ (log(log\ s_0))^{\frac{1}{2}}}$,

There are also singularities at

$$2\left[\frac{3}{|\theta(0)|}+9log(log\ s)\right]-\frac{\frac{3}{2}|\theta(0)|^2}{\left[\frac{1}{|\theta(0)|}+3log(log\ s)\right]}+... \quad (5.7)$$

$$s\approx\frac{3}{|\theta(0)|}+9log(log\ s)\pm\frac{3(6\ log(log\ s))^{\frac{1}{2}}}{|\theta(0)|^{\frac{1}{2}}}\sqrt{1+\frac{3log(log\ s)}{\frac{2}{|\theta(0)|}}} \quad (5.8)$$

when $9log(log\ s)\ll\frac{3}{|\theta(0)|}$, which can occur if $\frac{3}{|\theta(0)|}\gg 1$, and $|\theta(0)|\approx\frac{3}{s_0}$.

The existence of an equilibrium solution to the Fokker-Planck scale equation for the probabilistic evolution of the three-metric which decreases as $|\theta(0)|\to\infty$ would appear to be a prediction of geodesic completeness. The differential equation for the probability distribution is

$$\partial_s\mathcal{P}(\theta,s)=\partial_\theta[\alpha(\theta)\mathcal{P}(\theta,s)]+\frac{1}{2}\mu(a,b)\partial_\theta^2[\alpha(\theta)^2\mathcal{P}(\theta,s)] \quad (5.9)$$

where $\mu(a, b)$ is the diffusion coefficient between a and b and $\alpha(\theta)$ is the function in the stochastic generalization of the inequality derived from the Raychaudjuri equation. Extremization along the congruence of geodesics requires

$$\begin{aligned}\partial_\theta[\alpha(\theta)\mathcal{P}(\theta)]&=-\frac{1}{2}\mu(a,b)\partial_\theta^2(\alpha(\theta)^2\mathcal{P}(\theta))\\ -\frac{2}{\mu(a,b)\alpha(\theta)}&=\partial_\theta[ln(\alpha(\theta)^2\mathcal{P}(\theta))]\end{aligned} \quad (5.10)$$

with the solution

$$\mathcal{P}(\theta)=\frac{N}{\mu(a,b)\alpha(\theta)^2}exp\left(-\int^\theta\frac{2}{\mu(a,b)\alpha(\theta(0))}d\theta^2\right). \quad (5.11)$$

where N is the normalization factor. If $\alpha(\theta)=-\frac{1}{3}\theta^2$,

$$\mathcal{P}(|\theta|) = \frac{9N}{\mu(a,b)|\theta|^4} exp\left(-\frac{6}{\mu(a,b)|\theta|}\right) \quad (5.12)$$

is the equilbrium Ito distribution with a vanishing derivative at $|\theta| = \frac{3}{2\mu(a,b)}$. Given a congruence with a negative value of θ at a point p on the geodesic, it would tend to $-\frac{3}{2\mu(a,b)}$ for $\mu(a,b) > 0$ and the convergence of the geodesics does not occur since the probability for

However, if a caustic develops, the state of thermal equilibrium is not reached and it is not allowed to set $\frac{\partial}{\partial s} P(h_{ij}(\vec{x}, s))$ equal to zero. The problem of geodesic incompleteness may be resolved by a more general probabilistic evolution than that afforded by the Fokker-Planck equation. The method for finding the equation for θ therefore differs from the solution for the equilibrium probability distribution. The order of calculation is first-order for θ and secondorder for P(θ, s), with functional derivatives respect to the metric.

It may be noted that the asymptotic expansions of the wavefunction representing the probabilistic approximation of relativistic motion near caustics has been developed through solutions to comparable differential equations including the Airy function [6]. The equation for the expansion with no shear and rotation and sufficiently low curvature yields

$$\frac{d^2\theta}{ds^2} \approx -\frac{2}{3}\theta\frac{d\theta}{ds} \quad (5.13)$$

$$\theta \approx \frac{3}{s - s_0} \quad (5.14)$$

with the boundary condition $\theta \to -\infty$ as $s \to s_0$. Then $\theta(0) \approx -\frac{3}{s_0}$, $\frac{d\theta}{ds} \approx -\frac{3}{(s-s_0)^2}$ and

$$\frac{d^2\theta}{ds^2} - \frac{2}{(s-s_0)^2}\theta \approx 0 \quad (5.15)$$

near the caustic. If the stochastic term is included,

$$\frac{d\theta}{ds} \approx -\frac{3}{\left[\left(\frac{3}{|\theta(0)|}\right) - s + 3\int^s d\mathcal{B}(s)\right]^2} \left\{3\left(2\ (log(log\ s))^{\frac{1}{2}}\right) - 1 + \frac{1}{log\ s}(log(log\ s))^{-\frac{1}{2}}\right\} \quad (5.16)$$

and

$$\frac{d^2\theta}{ds^2} - \frac{2}{\left[\left(\frac{3}{|\theta(0)|}\right) - s + 3\int^s d\mathcal{B}(s)\right]^2}\left\{3\left(2\ (log(log\ s))^{\frac{1}{2}}\right) - 1 + \frac{1}{log\ s}(log(log\ s))^{-\frac{1}{2}}\right\}\theta \approx 0. \tag{5.17}$$

Instead of the singularities near the caustic, these equations may be replaced by an Airy equation

$$\frac{d^2\theta}{ds^2} + s\theta = 0 \tag{5.18}$$

with the regular solution representing a smooth variation about the geodesic trajectory of the congruence.

These solutions, therefore, are equivalent to the wavefunctions satisfying the Wheeler-DeWitt equation, which represent a regularization of a classical metric with intial curvature singularities.

The generalization of point particles to strings again can be described through the sum over histories. Equations for the convergence of congruences on string worldsheets have been given [9]. The basis $\{E_a^\mu, n_i^\mu, a = 1, ..., d,\ i = d+1, ..., D\}$ is a basis at each point on a timelike surface in an embedding space. Given that $D_a E_b = \gamma^c{}_{ab} E_c - K^i{}_{ab} n_i$, $D_a n^i = K^i{}_{ab} E^b + \omega_a^{ij} n_j$, $D_i E_a = J_{aij} n^j + S_{abi} E^b$, $D_i n_j = -J_{aij} E^a + \gamma_{ij}^k n_k$, and $J_a^{ij} = \Sigma_a^{ij} + \Lambda_a^{ij} + \frac{1}{D-d}\delta^{ij}\theta_a$,

$$\nabla_a \theta^a + \frac{1}{D-d}\theta_a\theta^a + (M^2)^i{}_i = 0. \tag{5.19}$$

Where

$$(M^2)^{ij} = K^i{}_{ab}K^{abj} + R_{\mu\nu\rho\sigma}E_a^\mu n^{\nu i}E^{\rho a}n^{\sigma j}. \tag{5.20}$$

Since $K^i = 0$ for extremal timelike Nambu-Goto surfaces,

$$\nabla_a \theta^a + \frac{1}{8}\theta_a\theta^a + (M^2)^i{}_i = \nabla_a \theta^a + \frac{1}{8}\theta_a\theta^a + R_{\mu\nu\rho\sigma}E_a^\mu n^{\nu i}E^{\rho a}n_{\sigma i} = 0. \tag{5.21}$$

For a one-dimensional curve in four dimensions, with $\nabla_a \theta^a$ replaced by $\frac{d\theta}{ds}$ and $\theta_a\theta^a = \theta^2$,

$$\frac{d\theta}{ds} + \frac{1}{3}\theta^2 \leq 0 \tag{5.22}$$

when the strong energy condition condition $R_{\mu\nu\rho\sigma}t^{\mu}n^{i\nu}t^{\rho}n^{i\sigma} = R_{\mu\rho}t^{\mu}t^{\rho} \geq 0$ Is satisfied by the components of the Ricci tensor.

A surface of genus greater than or equal to two may be given a hyperbolic metric with a curvature equal to -1. If $\nabla_a\theta^a \geq 0$ initially, the effect of $-\frac{1}{3}\theta_a\theta^a$ will be the decrease of θuntil there is a cancellation of $-\frac{1}{3}\theta_a\theta^a$ and $R_{\mu\rho}E^{\mu}_{a}E^{a\rho} = {}^{(2)}R$, and |θa| → 0. Given a surface with a minimum of two handles, the geodesic congruences may be continued without caustics to the end, which may be located at an arbitrarily large distance. By constrast, the geodesic congruences on a surface with no handles or one handles will develop caustics.

It may be noted that the sum over surfaces can be defined formally as the domain of string perturbation theory. A study of this class of surfaces reveals that the series can be extended to infinite genus. Although the infinite-genus surfaces might be required to be confined to a finite interaction region, an affine parameter on curves on surfaces of this type may be shown to reach ∞ in the intrinsic metric because the intrinsic area is infinite. Furthermore, a curve in the target space-time with an infinite range for the affine parameter would exist on a union of Riemann surfaces.

6. CONCLUSION

It appears that the introduction of quantum effects on paths in space-time has depends essentially on the mechanism for producing variations about the classical geodesics. In addition to the previous analysis of the distribution of three-metrics determined by the Fokker-Planck equation, the stochastic quantization of the motion of perturbations about classical trajectories.

If the stochastic variations are given by by $\vec{x}_t(\epsilon) = \vec{x}_t(0) + \epsilon\vec{v}(\vec{x}_t(0), t)$ then the lengths of the quantum trajectories have been found to be $L(x_0^t) \sim v_0 t + \hbar\left(\frac{1}{4v_0} arc\ tan\left(\frac{t}{A}\right)\right) + \mathcal{O}(\hbar^2)$

A determines the initial packet width through $\psi(x,0) = (\pi\hbar A)^{-\frac{1}{4}} exp\left(-\frac{x^2}{2\hbar A}\right)$ $\cdot \exp\left(i\frac{v_0}{\hbar}x\right)$ [7]. As $A \to \infty$, the spread in the initial wave packet increases, and, since ~A equals the variance in a normal distribution, the length of the quantum trajectory is not significantly increased. When $A \rightarrow 0$, the spread in the initial wave packet decreases, and the length of the quantum trajectory increases.

The counteracting effects imply that the spread of the wave packet satisfying the constraints imposed by stochastic quantization would not yield variations that are arbitrarily far from the classical geodesics. Consequently, it is possible that singularities in the trajectories arise before the affine parameter reaches infinity as there is not a sufficient separation from the caustics to avoid the effect of the curvature of the space-time.

It has been suggested, alternatively, under more general circumstances, that the spread in the wave packet can be quantized and that it yields a standard wave equation [8]. The increase in the variation of the quantum trajectories with respect to the affine parameter would cancel the converging effect of focussing of congruences.

An further development of this approach is evident in the path integral which contains arbitrary smooth paths with a weighting factor given by iI[C] where I[C] is the action of the curve. Since the variations do not seem to be constrained to be located in a specified region surrounding the classical trajectories, it would appear to be possible to find quantum trajectories sufficiently different from the geodesics such that the caustics could be avoided. Since this could be done an arbitrarily large number of times, a potential resolution to the problem of geodesic incompleteness of a space-time satisfying the dominant energy condition then exists.

It follows, therefore, that the existence of quantum trajectories with infinite pathlength on space-times could be proven if the quantum variations about the classical trajectories are not overly constrained. This appears to be possible in a variety of approaches both in point-particle and string theories.

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