ON COMPUTATIONS OF THPD MATRICES

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ABSTRACT

Toeplitz Hermitian Positive Definite (THPD) matrices play an important role in signal processing and computer graphics and circular models, related to angular / periodic data, have vide applications in various walks of life. Visualizing a circular model through THPD matrix the required computations on THPD matrices using single bordering and double bordering are discussed. It can be seen that every tridiagonal THPD leads to Cardioid model.

1. INTRODUCTION

Every Toeplitz Hermitian Positive Definite matrix (THPD) can be associated to a Circular model through its characteristic function [Mardia (1972)]. The idea of THPD matrix, a special case of Toeplitz matrix has a natural extension to infinite case as well. Here, it is attempted to present certain algorithms for computations on THPD matrices.

Section 2 is devoted to explain the association between Circular models and THPD matrices which is the motivation for taking up computations on THPD matrices for possible future applications / extensions. On the lines of Rami Reddy (2005), methods for LU decomposition of a THPD matrix using single bordering algorithms are discussed and are presented in Sections 3.

2. REPRESENTATION OF A CIRCULAR MODEL THROUGH THPD MATRIX

In continuous case the probability density function (pdf) $g(\theta)$ of a circular distribution exists and has the following basic properties

- $g(\theta) \geq 0, \ \forall\theta$ (2.1)

- $\int_0^{2\pi} g(\theta)\, d\theta = 1$ (2.2)

- $g(\theta) = g(\theta + 2k\pi)$ for any integer k (i.e., g is periodic) (2.3)

Rao and Sengupta (2001).

Distribution function G on the circle is uniquely determined by characteristic function [c.f. p. 80 Mardia (1972)]. The characteristic function and cdf of the resultant wrapped circular model are,

$$\phi_W(p) = E(e^{ip\theta}) = \phi(p), \quad p = 0, \ \pm 1, \ \pm 2, \ \pm 3, ...$$

and $G(\theta) = \int_0^{\theta} g(\theta') d\theta', \theta \in [0, 2\pi)$

Consider the sequence $\{\phi_p\}_{-\infty}^{\infty}$ associated with a characteristic function ϕ of a circular distribution. It is known that if $\sum_{p=-\infty}^{\infty} |\phi_p|^2$ is convergent then $\{\phi_p\} \in l^2$ and $\sum_{p=-\infty}^{\infty} |\phi_p|^2$ is convergent iff $\sum_{p=0}^{\infty} |\phi_p|^2$ is convergent. Thus the sequence $\{\phi_p\}$ of a circular model satisfies (i) $\phi_0 = 1$, (ii) $\phi_{-p} = \bar{\phi}_p$, $\sum_{-\infty}^{\infty} |\phi_p|^2 < \infty$. Since $\lim \phi_p = 0$, one can assume that $|\phi_p| < 1 \ \ \forall \ p$.

If $\phi_p = \alpha_p + i\beta_p$, then the pdf of the corresponding Circular model is given by

$$g(\theta) = \frac{1}{2\pi} \left[1 + 2\sum_{p=1}^{\infty} \left(\alpha_p \cos p\theta + \beta_p \sin p\theta \right) \right] \tag{2.4}$$

Further if $g(\theta)$ is a function of bounded variation then the Fourier series $g(\theta) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \phi_p e^{-ip\theta}$ converges [Jordan's test] provided

1. $g(\theta) = \frac{1}{2}\{g(\theta+0) + g(\theta-0)\}$ or
2. $g(\theta)$ has only a finite number of maxima and minima and a finite number of discontinuities in the interval $[0, 2\pi)$ and it can be proved that $\phi_p = E(e^{ip\theta}) = \int_0^{2\pi} e^{ip\theta} \, d(G(\theta)).$

Thus the characteristic function of a Circular distribution G is represented by the sequence $\{\phi_p\}$.

Let the matrix $A = (a_{ij})$ where $a_{ij} = \phi_{i-j}$, $i, j \geq 0$ and $a_{ij} = \bar{a}_{ji}$. Since $\phi_{-p} = \bar{\phi}_p$ and all leading principal minors of A are positive, therefore, A is positive definite.

Thus by invoking the Inversion theorem for characteristic functions it follows that for every Circular model there corresponds an infinite matrix through it's characteristic function. Further, all the leading principal minors are positive, hence the matrix A is THPD matrix. If $\alpha_p = \beta_p = 0$ for $p \geq n$, i.e.

$g(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{n}\left(\alpha_p \cos p\theta + \beta_p \sin p\theta\right)\right]$, then the corresponding matrix becomes finite THPD matrix.

By definition a matrix A is tridiagonal from Girija (2004) if $a_{ij} = 0$ whenever $|i - j| \geq 2$. When A is a tridiagonal THPD, then as shown below A induces a Cardioid distribution for appropriate parameters. In view of the involvement of tridiagonal matrices in Circular distributions particularly in Cardioid distribution we may as well look into the properties preserved by the tridiagonal matrices in general. We briefly present a few computational methods connected with tridiagonal matrices.

2.1Cardioid distribution and tridiagonal matrices

The pdf of a Cardioid distribution with parameters μ and ρ is given by

$$g(\theta) = \frac{1}{2\pi}\left(1 + 2\rho\cos(\theta - \mu)\right), \ \theta \in [0, 2\pi), \ -\frac{1}{2} < \rho < \frac{1}{2}$$

The trigonometric moments of g are

$$\alpha_1 = 2\rho\cos\mu \ , \ \beta_1 = 2\rho\sin\mu .$$

The corresponding characteristic function of the Cardioid distribution is

$$\phi_W(p) = \alpha_p + i\beta_p, \ p = 0, \pm 1.$$

Hence, the THPD matrix induced by the characteristic function of a Cardioid distribution is

$$A = \begin{pmatrix} \phi_0 & \bar{\phi}_1 \\ \phi_1 & \phi_0 \end{pmatrix}$$

Clearly for any $n \geq 2$, the tridiagonal THPD matrix of order n

$$A_n = \begin{pmatrix} \phi_0 & \bar{\phi}_1 & 0 & \dots & 0 \\ \phi_1 & \phi_0 & \bar{\phi}_1 & \dots & 0 \\ 0 & \phi_1 & \phi_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \phi_1 & \phi_0 \end{pmatrix}$$

represents a Cardioid distribution. However the minimum order of such matrix is 2.

3. LU – DECOMPOSITION OF A TOEPLITZ POSITIVE DEFINITE MATRIX BY SINGLE BORDERING

Unless otherwise specified, the matrices mentioned here by default are square matrices of order n. By a LU-decomposition of a matrix A we mean a decomposition of A as $A = LU$ where L and U are lower triangular matrices respectively. This decomposition is possible [Jain and Chawla (1971)] when the leading submatrices of A are nonsingular.

NOTE :

1. A matrix may not have LU – decomposition as is evident from the following.

Example 1 :

$$\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} l_1 & 0 \\ l_2 & l_3 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}$$

$\Rightarrow l_1 u_1 = 0, l_2\, u_1 = 3,\ l_1 u_2 = 2$ which is impossible.

2. Even if LU decomposition exists, it may not be unique.

Example 2.

$$\begin{pmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ -3 & 6 & -2 \end{pmatrix}\begin{pmatrix} -1 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 2 & -4 & 0 \\ 3 & -6 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

3. It is known that the LU decomposition is unique when all the entries on the diagonal of L or U are unity.

Having established the association between a THPD matrices and Circular models, the following computational aspects on THPD matrix are explored.

LU- decomposition of a THPD matrix by single bordering.

Theorem 3.2

Let A be a THPD matrix of order n and have the form

$A = \begin{pmatrix} A_{n-1} & x_{n-1} \\ x^*_{n-1} & a_0 \end{pmatrix}$ where A_{n-1} is a nonsingular matrix of order (n-1) and $x^*_{n-1} = \begin{pmatrix} a_{n-1} & a_{n-2} & \ldots & \ldots & a_1 \end{pmatrix}$.

Assume that

LU-decomposition of A_{n-1} exists and is given by $A_{n-1} = L_{n-1}U_{n-1}$, then $A = LU$ where

$$L = \begin{pmatrix} L_{n-1} & O \\ l_{n-1}' & l_{nn} \end{pmatrix} \text{ and } U = \begin{pmatrix} U_{n-1} & u_{n-1} \\ O & 1 \end{pmatrix}$$

$$u_{n-1} = L_{n-1}{}^{-1}x_{n-1}, l_{n-1}' = x^*_{n-1}U_{n-1}{}^{-1} \text{ , } l_{nn} = a_0 - l_{n-1}'u_{n-1}$$

Proof: Since A_{n-1} is THPD matrix $A_{n-1}{}^{-1}$ exists, hence L_{n-1}, U_{n-1} are invertible.

$$LU = \begin{pmatrix} L_{n-1} & O \\ l_{n-1}' & l_{nn} \end{pmatrix}\begin{pmatrix} U_{n-1} & u_{n-1} \\ O & 1 \end{pmatrix} = \begin{pmatrix} L_{n-1}U_{n-1} & L_{n-1}u_{n-1} \\ l_{n-1}'U_{n-1} & l_{n-1}'u_{n-1} + l_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} A_{n-1} & x_{n-1} \\ x^*_{n-1} & a_0 \end{pmatrix} = A$$

$$\Leftrightarrow L_{n-1}U_{n-1} = A_{n-1} \text{ , } L_{n-1}u_{n-1} = x_{n-1} \text{ , } l_{n-1}'U_{n-1} = x^*_{n-1} \text{ ,}$$
$$l_{n-1}'u_{n-1} + l_{nn} = a_0$$

$$\Leftrightarrow l_{nn} = a_0 - x_{n-1}'U_{n-1}{}^{-1}L_{n-1}{}^{-1}x_{-(n-1)} = a_0 - x_{n-1}'A_{n-1}{}^{-1}x_{-(n-1)}$$

$$\Leftrightarrow x^*_{n-1} = l^T_{n-1}U^{-1}_{n-1} \Leftrightarrow x_{n-1} = \left(U_{n-1}\right)^* l_{n-1}$$

$$u_{n-1} = L^{-1}_{n-1}\left(U_{n-1}\right)^* l_{n-1}$$

$$u_{n-1} = L^{-1}_{n-1}\left(U^*_{n-1}\right) l_{n-1}$$

For the purpose of computations, the above theorem is arranged in the form of a Recursive algorithm.

Given $A = (a_{i,j})$

Let $a_{-i} = \overline{a}_i$

Step 1 : $A_1 = (a_0)$, $L_1 = (a_0)$, $U_1 = (1)$

Step 2 : For $i = 2\ (1)\ n$ we have

and $x_{i-1}^{*} = (a_{i-1}, a_{i-2}, \ldots a_1)$

Compute L_{i-1}^{-1}, U_{i-1}^{-1}

Write $l_{i-1}^{T} = x_{i-1}^{*} U_{i-1}^{-1}$, $u_{i-1} = L_{i-1}^{-1} x_{i-1}$

Compute $l_{i-1}^{T} u_{i-1}$

Write $l_{ii} = a_0 - l_{i-1}^{T} u_{i-1}$

Write $L_i = \begin{pmatrix} L_{i-1} & 0 \\ l_{i-1}^{T} & l_{ii} \end{pmatrix}$, $U_i = \begin{pmatrix} U_{i-1} & u_{i-1} \\ 0 & 1 \end{pmatrix}$

Write $L = L_n$, $U = U_n$

Example

We find the LU decomposition for the following THPD matrix A is associated with the characteristic function of a Wrapped Weibull distribution.

$$A = \begin{pmatrix} 1.0000 & 0.5432+0.6347i & 0.0276+0.5312i & -0.1012+0.2970i \\ 0.5432-0.6347i & 1.0000 & 0.5432+0.6347i & 0.0276+0.5312i \\ 0.0276-0.5312i & 0.5432-0.6347i & 1.0000 & 0.5432+0.6347i \\ -0.1012-0.2970i & 0.0276-0.5312i & 0.5432-0.6347i & 1.0000 \end{pmatrix}$$

Step1 :

$A_1 = (1)$, $L_1 = (1)$, $U_1 = (1)$

Step 2:

x_1 = [0.5432 - 0.6347i]

$L_1^{-1} = (1)$, $U_1^{-1} = (1)$

l_1 = 0.5432 - 0.6347i

u_1 = 0.5432 + 0.6347i

l_{22} = 0.3021

$$L_2 = \begin{pmatrix} 1.0000 & 0 \\ 0.5432 - 0.6347\mathrm{i} & 0.3021 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 1.0000 & 0.5432 + 0.6347i \\ 0 & 1.0000 \end{pmatrix}$$

$$L_2\, U_2 = \begin{pmatrix} 1.0000 & 0.5432 + 0.6347i \\ 0.5432 - 0.6347i & 1.0000 \end{pmatrix}$$

Step 3:

$$x_2 = [0.0276 - 0.5312i \quad 0.5432 - 0.6347i]$$

$$L_2^{-1} = \begin{pmatrix} 1.0000 - 0.0000i & 0 \\ -1.7981 + 2.1010i & 3.3103 \end{pmatrix}$$

$$U_2^{-1} = \begin{pmatrix} 1.0000 & -0.5432 - 0.6347i \\ 0 & 1.0000 \end{pmatrix}$$

$$l_2 = (\ 0.0276 - 0.5312i \quad 0.1911 - 0.3637i)$$

$$u_2 = \begin{pmatrix} 0.0276 + 0.5312i \\ 0.6324 + 1.2038i \end{pmatrix}$$

$$l_{33} = (\ 0.1584 - 0.0000i\ )$$

$$L_3 = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0.5432 - 0.6347i & 0.3021 & 0 \\ 0.0276 - 0.5312i & 0.1911 - 0.3637i & 0.1584 - 0.0000i \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 1.0000 & 0.5432 + 0.6347i & 0.0276 + 0.5312i \\ 0 & 1.0000 & 0.6324 + 1.2038i \\ 0 & 0 & 1.0000 \end{pmatrix}$$

$$L_3 U_3 = \begin{pmatrix} 1.0000 & 0.5432 + 0.6347i & 0.0276 + 0.5312i \\ 0.5432 - 0.6347i & 1.0000 & 0.5432 + 0.6347i \\ 0.0276 - 0.5312i & 0.5432 - 0.6347i & 1.0000 \end{pmatrix}$$

Step 4:

x_3 = [-0.1012 - 0.2970i 0.0276 - 0.5312i 0.5432 - 0.6347i]

$$L_3^{-1} = \begin{pmatrix} 1.0000 - 0.0000i & 0 & 0 \\ -1.7981 + 2.1010i & 3.3103 & 0 \\ -2.8286 - 3.3084i & -3.9920 + 7.5986i & 6.3119 + 0.0000i \end{pmatrix}$$

$$U_3^{-1} = \begin{pmatrix} 1.0000 & -0.5432 - 0.6347i & -0.4481 + 0.5241i \\ 0 & 1.0000 & -0.6324 - 1.2038i \\ 0 & 0 & 1.0000 \end{pmatrix}$$

l_3 = [-0.1012 - 0.2970i -0.1059 - 0.3056i 0.0873 - 0.2519i]

$$u_3 = \begin{pmatrix} -0.1012 + 0.2970i \\ -0.3507 + 1.0117i \\ 0.5509 + 1.5901i \end{pmatrix}$$

l_{44} = (0.1065 - 0.0000i)

$$L_4 = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.5432 - 0.6347i & 0.3021 & 0 & 0 \\ 0.0276 - 0.5312i & 0.1911 - 0.3637i & 0.1584 - 0.0000i & 0 \\ -0.1012 - 0.2970i & -0.1059 - 0.3056i & 0.0873 - 0.2519i & 0.1065 - 0.0000i \end{pmatrix}$$

$$U_4 = \begin{pmatrix} 1.0000 & 0.5432 + 0.6347i & 0.0276 + 0.5312i & -0.1012 + 0.2970i \\ 0 & 1.0000 & 0.6324 + 1.2038i & -0.3507 + 1.0117i \\ 0 & 0 & 1.0000 & 0.5509 + 1.5901i \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

$$L_4\,U_4 = \begin{pmatrix} 1.0000 & 0.5432+0.6347\mathrm{i} & 0.0276+0.5312\mathrm{i} & -0.1012+0.2970\mathrm{i} \\ 0.5432-0.6347\mathrm{i} & 1.0000 & 0.5432+0.6347\mathrm{i} & 0.0276+0.5312\mathrm{i} \\ 0.0276-0.5312\mathrm{i} & 0.5432-0.6347\mathrm{i} & 1.0000 & 0.5432+0.6347\mathrm{i} \\ -0.1012-0.2970\mathrm{i} & 0.0276-0.5312\mathrm{i} & 0.5432-0.6347\mathrm{i} & 1.0000 \end{pmatrix}$$

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