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Stereographic Circular Normal Moment Distribution

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ABSTRACT

Minh et al (2003) and Toshihiro Abe et al (2010) proposed a new method to derive circular distributions from the existing linear models by applying Inverse stereographic projection or equivalently bilinear transformation. In this paper, a new circular model, we call it as stereographic circular normal moment distribution, is derived by inducing modified inverse stereographic projection on normal moment distribution (Akin Olosunde et al (2008)) on real line. This distribution generalizes stereographic circular normal distribution (Toshihiro Abe et al (2010)), the density and distribution functions of proposed model admit closed form. We provide explicit expressions for trigonometric moments.

KEYWORDS

characteristic function, circular models, Inverse stereographic projection, shape parameter, trigonometric moments.

1. INTRODUCTION

Directions in two-dimensions can be represented as points on the circumference of a unit circle and models for representing such data are called circular distributions. Quite a lot of work was done on circular models defined on the unit circle (Fisher, 1993; Jammalamadaka and Sen Gupta (2001); Mardia and Jupp, (2000)) and recent publications (Dattatreya Rao et al (2007), Girija (2010), Phani et al (2012)). Normal moment distribution is a particular case of well known Kotz-type elliptical distribution. Akin Olosunde et al. (2008) given a short survey concerning Normal Moment Distribution. The aim of the present article is to construct a circular model. In this paper, we use Modified Inverse Stereographic Projection to define a new circular model, so called The Stereographic Circular Normal Moment Distribution which generalizes the Stereographic Circular Normal Distribution (Toshihiro Abe et al (2010)). We provide explicit expressions for trigonometric moments.

The rest of the paper is organized as follows: in section 2, we present methodology of modified inverse stereographic projection. In section 3, we introduce the proposed distribution and present graphs of probability density function for various values of parameters. In section 4, we derive the first two trigonometric moments. In section 5, our work is concluded.

2. Methodology of Modified Inverse Stereographic Projection

Modified Inverse Stereographic Projection is defined by an one to one mapping given by $\tau(\theta) = x = \nu \tan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty), \theta \in [-\pi, \pi), \nu \in \mathbb{R}^+$. Suppose x is randomly chosen on the interval $(-\infty, \infty)$. Let $F(x)$ and $f(x)$ denote the cumulative distribution and probability density functions of the random variable X respectively. Then $\tau^{-1}(x) = \theta = 2\tan^{-1}\left(\frac{x}{\nu}\right)$ is a random point according to Toshihiro Abe et al (2010) on the unit circle. Let $G(\theta)$ and $g(\theta)$ denote the cumulative distribution and probability density functions of this random point θ respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following Theorem.

Theorem 2.1: For $\nu > 0$,

i) $$G(\theta) = F\left(\nu \tan\left(\frac{\theta}{2}\right)\right)$$

ii) $$g(\theta) = \nu\left(\frac{\sec^2\left(\frac{\theta}{2}\right)}{2}\right) f\left(\nu \tan\left(\frac{\theta}{2}\right)\right)$$

If a linear random variable X has a support on $\mathbb{R}$, then θ has a support on $(-\pi, \pi)$. These means that, after the Inverse stereographic projection is applied, we can deal circular data if the support of X is on $\mathbb{R}$. Throughout this paper we assume $\nu = 1$.

3. Stereographic Normal Moment Distribution

Here we recall a probability density and distribution functions of Normal Moment Distribution (Akin Olosunde [1]).

A random variable X on the real line is said to follow Normal Moment distribution with shape parameter $\alpha \geq 0$, if the probability density function and cumulative distribution function of X are respectively given by

$$f(x) = \frac{1}{2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)}(x)^{2\alpha} e^{-\frac{(x)^2}{2}}, \quad -\infty < x < \infty \ , \ \alpha \geq 0 \tag{3.1}$$

$$F(x)=\frac{1}{\Gamma\left(\alpha+\frac{1}{2}\right)}\Gamma_x\left(\alpha+\frac{1}{2}\right) \tag{3.2}$$

where

$\Gamma_x\left(\alpha+\frac{1}{2}\right)$ is a Incomplete Gamma Function.

Now we can induce modified inverse stereographic projection on normal moment distribution, which leads to a circular distribution on unit circle. We call it as Stereographic Circular Normal Moment.

Definition:

A random variable θ on unit circle is said to follows stereographic circular normal moment distribution with location parameter μ, scale parameter $\sigma>0$ and shape parameter $\alpha\geq 0$ denoted by SCNMD(μ,σ,α), if the probability density and cumulative distribution functions are respectively given by

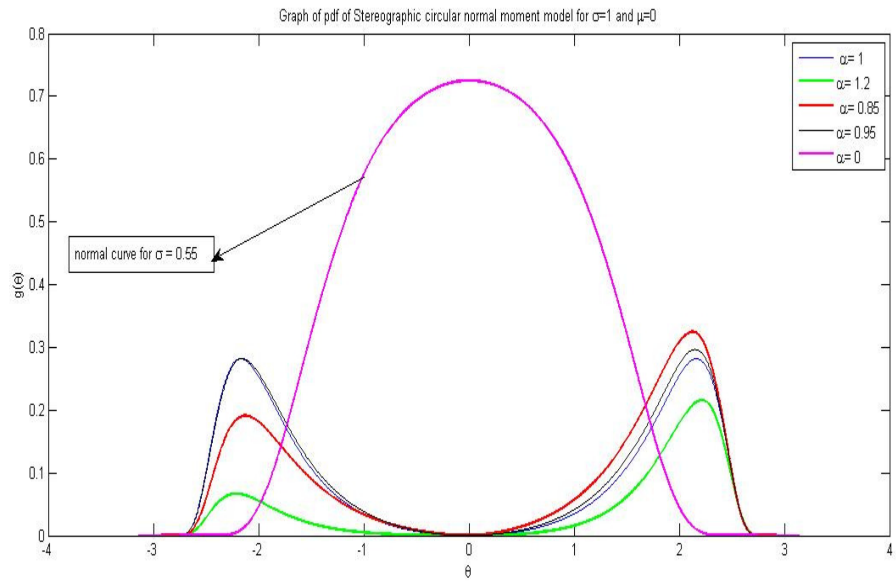
$$g(\theta)=\frac{1}{2\sigma 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sec^2\left(\frac{\theta-\mu}{2}\right)\left(\frac{1}{\sigma}\tan\left(\frac{\theta-\mu}{2}\right)\right)^{2\alpha}\exp\left(-\frac{1}{2}\left(\frac{\tan\left(\frac{\theta-\mu}{2}\right)}{\sigma}\right)^2\right),$$

$$\text{Where } \alpha\geq 0,\ \sigma>0 \text{ and } -\pi\leq\theta,\ \mu<\pi \tag{3.3}$$

$$G(\theta)=\frac{\Gamma_{\left(\tan\left(\frac{\theta-\mu}{2}\right)\right)}\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}, \tag{3.4}$$

It can be seen that the Stereographic Circular Normal Moment Distribution is Symmetric when $\mu=0$ and it generalizes Stereographic Normal Distribution for $\alpha=0$.

Graphs of probability density function of Stereographic Circular Normal Moment Distribution for various values of σ , α and $\mu=0$ *are presented here.*



4.The Characteristic Function Of Stereographic Model

The Characteristic function of a Circular model with probability density function $g(\theta)$ is defined as $\varphi_p(\theta) = \int_0^{2\pi} e^{ip\theta} g(\theta)\, d\theta$, $p \in \mathbb{R}$.

Theorem 4.1 : If $G(\theta)$ and $g(\theta)$ are the cumulative distribution function and probability density function of the stereographic circular model and $F(x)$ and $f(x)$ are cumulative distribution function and probability density function of the respective linear model, then characteristic function of stereographic model is $\varphi_{X_S}(p) = \varphi_{2\tan^{-1}(x)}(p), p \in \mathbb{R}$

Proof:

$$
\begin{aligned}
\varphi_{X_S}(p) &= \int_{-\pi}^{\pi} e^{ip\theta} d\left(G(\theta)\right) \quad , p \in \mathbb{R} \\
&= \int_{-\pi}^{\pi} e^{ip\theta} d\left(F\left(\tan\frac{\theta}{2}\right)\right) \\
&= \int_{-\infty}^{\infty} e^{ip\left(2\tan^{-1}(x)\right)} f(x)\, dx, \quad \text{taking } x = \tan\left(\frac{\theta}{2}\right) \\
&= \varphi_{2\tan^{-1}(x)}(p)
\end{aligned}
$$

The Characteristic function of Stereograph Circular Normal Moment Model:

$$\varphi_\theta(p)=\int_{-\pi}^{\pi} e^{ip\theta} g(\theta)\,d\theta$$

$$=\frac{2}{\sigma 2^{\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\infty \cos\left(2\tan^{-1}(x)\right)x^{2\alpha}e^{-\left(\frac{x^2}{2\sigma^2}\right)}dx$$

As the integral cannot admits closed form, MATLAB techniques are applied for the evaluation of the values of the characteristic function. For computing population characteristics, the first two moments are sufficient, so we present below the first two moments.

Without loss of generality, we assume that $\mu=0$ in SCNMD. The trigonometric moments of the distribution are given by $\{\varphi_{\mathrm{p}}: \mathrm{p}=\pm1,\pm2,\pm3,...\}$, where $\varphi_{\mathrm{p}}=\alpha_{\mathrm{p}}+i\beta_{\mathrm{p}}$, with $\alpha_{\mathrm{p}}=E(\cos \mathrm{p}\theta)$ and $\beta_{\mathrm{p}}=E(\sin \mathrm{p}\theta)$ being the p^{th} order cosine and sine moments of the random angle θ, respectively. Because the stereographic circular normal moment distribution is symmetric about $\mu=0$, it follows that the sine moments are zero. Thus, $\varphi_{\mathrm{p}}=\alpha_{\mathrm{p}}$.

Theorem 4.2: Under the pdf of the stereographic circular normal moment distribution with $\mu=0$, the first two $\alpha_{\mathrm{p}}=E(\cos \mathrm{p}\theta)$, $\mathrm{p}=1,2$ are given as follows

$$\alpha_1=1-\frac{1}{2^{\left(\alpha-\frac{1}{2}\right)}\sigma^{(2\alpha+1)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{n=0}^{\infty}(-1)^n\left(2\sigma^2\right)^{\left(\frac{2n+2\alpha+3}{2}\right)}\Gamma\left(\frac{2n+2\alpha+3}{2}\right), \tag{4.1}$$

$$\alpha_2=1+\frac{4}{2^{\left(\alpha-\frac{1}{2}\right)}\sigma^{(2\alpha+1)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(2\sigma^2\right)^{\left(\frac{2n+2\alpha+5}{2}\right)}\Gamma\left(\frac{2n+2\alpha+5}{2}\right)$$

$$-\frac{4}{2^{\left(\alpha-\frac{1}{2}\right)}\sigma^{(2\alpha+1)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(2\sigma^2\right)^{\left(\frac{2n+2\alpha+3}{2}\right)}\Gamma\left(\frac{2n+2\alpha+3}{2}\right) \tag{4.2}$$

Where $\int_0^\infty x^{\nu-1}e^{-\mu x^p}dx=\frac{1}{p}\mu^{-\left(\frac{\nu}{p}\right)}\Gamma\left(\frac{\nu}{p}\right),$ **(4.3)**

for $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$ and $\mathrm{p} > 0$ **(Gradshteyn and Ryzhik, 2007, formula no. 3.478.1).**

Proof: $\varphi_{\mathrm{p}} = \int_{-\pi}^{\pi} \cos(\mathrm{p}\theta) g(\theta)\, d\theta = \alpha_{\mathrm{p}}$

Where $$\alpha_{\mathrm{p}} = \frac{1}{2\sigma 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-\pi}^{\pi} \cos \mathrm{p}\theta \sec^2\left(\frac{\theta}{2}\right)\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^{2\alpha} e^{-\frac{1}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^2} d\theta$$

We derive the first order trigonometric moment

$$\alpha_1 = \frac{1}{2\sigma 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-\pi}^{\pi} \cos\theta \sec^2\left(\frac{\theta}{2}\right)\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^{2\alpha} e^{-\frac{1}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^2} d\theta \text{ as follows}$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos\theta = 1 - \frac{2x^2}{1+x^2}$ and the above formula **(4.3)**

$$\alpha_1 = \frac{1}{\sigma^{(2\alpha+1)} 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left[1 - \frac{2x^2}{1+x^2}\right]\left(x^{2\alpha}\right) e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= 1 - \frac{4}{\sigma^{(2\alpha+1)} 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} x^{(2n+2\alpha+3)-1} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$\alpha_1 = 1 - \frac{1}{\sigma^{(2\alpha+1)} 2^{\left(\alpha-\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \sum_{n=0}^{\infty} (-1)^n \left(2\sigma^2\right)^{\left(\frac{(2n+2\alpha+3)}{2}\right)} \Gamma\left(\frac{2n+2\alpha+3}{2}\right)$$

We derive the **second cosine moment**

$$\alpha_2 = \frac{1}{2\sigma 2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-\pi}^{\pi} \cos 2\theta \sec^2\left(\frac{\theta}{2}\right)\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^{2\alpha} e^{-\frac{1}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sigma}\right)^2} d\theta \text{ as follows}$$

Consider the transformation $x=\tan\left(\frac{\theta}{2}\right)$, $\cos 2\theta = 1+\frac{8x^4}{\left(1+x^2\right)^2}-\frac{8x^2}{\left(1+x^2\right)}$ and

the above formula **(4.3)**

$$\alpha_2 = \frac{1}{\sigma^{(2\alpha+1)}2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\left[1+\frac{8x^4}{\left(1+x^2\right)^2}-\frac{8x^2}{\left(1+x^2\right)}\right]\left(x^{2\alpha}\right)e^{-\frac{x^2}{2\sigma^2}}dx$$

$$=1+\frac{16}{\sigma^{(2\alpha+1)}2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{0}^{\infty}\frac{x^4}{\left(1+x^2\right)^2}\left(x^{2\alpha}\right)e^{-\frac{x^2}{2\sigma^2}}dx$$

$$-\frac{16}{\sigma^{(2\alpha+1)}2^{\left(\alpha+\frac{1}{2}\right)}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{0}^{\infty}\frac{x^2}{\left(1+x^2\right)}\left(x^{2\alpha}\right)e^{-\frac{x^2}{2\sigma^2}}dx$$

$$\alpha_2 = 1+\frac{4}{2^{\left(\alpha-\frac{1}{2}\right)}\sigma^{(2\alpha+1)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(2\sigma^2\right)^{\left(\frac{2n+2\alpha+5}{2}\right)}\Gamma\left(\frac{2n+2\alpha+5}{2}\right)$$

$$-\frac{4}{2^{\left(\alpha-\frac{1}{2}\right)}\sigma^{(2\alpha+1)}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(2\sigma^2\right)^{\left(\frac{2n+2\alpha+3}{2}\right)}\Gamma\left(\frac{2n+2\alpha+3}{2}\right)$$

by similar process, higher-order moments can be obtained.

5. Conclusion

In this paper, we investigated the circular distribution which follows from inducing modified inverse stereographic projection on normal moment distribution. The density and distribution function of stereographic circular normal moment distribution admit explicit forms, as do trigonometric moments. As this distribution is symmetric, promising for modeling symmetrical directional data.

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