On Ideals via Generalized Reverse Derivation On Factor Rings

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Abstract. In current article, for a prime ideal P of any ring R, we study the commutativity of the factor ring R/P, whenever R equipped with generalized reverse derivations F and G associated with reverse derivations d and g, respectively. That satisfies certain differential identities involving in P that connected to an ideal of R. Additionally, we show that, for some cases, the range of the generalized reverse derivation F or G repose in the prime ideal P. Moreover, we explore several consequences and special cases. Throughout, we provide examples to demonstrate that various restrictions in the assumptions of our outcomes are essential.

Keywords: prime ideal; integral domain; generalized reverse derivation; factor ring.

1 Introduction

In current article, R is an associative ring with center $Z(R)$. R is prime ring if and only if the set $\{0\}$ is prime ideal of R. On other words, R is a prime ring if $xRy = \{0\}$ then $x = 0$ or $y = 0$. P is a prime ideal of R if $P \neq R$ and for any two ideals I and J of R such that $IJ \subseteq P$, then one at least of I or J is involving in a prime ideal P.
An additive mapping $d : R \longrightarrow R$ is called a derivation if that satisfied the function $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation associated with the derivation d if the function $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. One of most famous examples of derivation is $d_m = [m, x]$ for any $x \in R$, which is called the inner derivation prompted by m. For a non-trivial example of a generalized derivation on a non-commutative ring, the interesting reader can refer to [1] and [2].
The concept of a reverse derivation was initially defined by Herstein in [3] when he had been proved that the prime ring R is a commutative integral domain whenever the imposed derivation is a Jordan derivation. It had been defined to be an additive mapping $d : R \longrightarrow R$ that satisfies the function $d(xy) = d(y)x + yd(x)$ for any $x, y \in R$. Several studies in reverse derivation field are appeared that studied the commutativity of a ring R; prime, semiprime or arbitrary ring. Samman et al. in [4] explored the reverse derivations in semiprime rings. In [5], Aboubakr et al. studied the relationship between generalized reverse derivations and generalized derivations in semiprime rings. A generalized reverse derivation associated with reverse derivation d is defined as an additive mapping $F : R \longrightarrow R$ satisfied the function $F(xy) = F(y)x + yd(x)$ for all $x, y \in R$. In case R is commutative ring, then generalized derivations and generalized reverse derivations is coincide. However, the converse does not always hold, as illustrated [[6], Example 1]. In [7], Ibraheem in its paper proved the prime ring commutativity under influences of generalized reverse derivation that associated with derivation d such that $[F(x), x]$ contained in a center of a ring, for all element x in a right ideal I of a ring R, provided that the intersection of a right ideal I and the center $Z(R)$ does not equal to zero. In [8] Bulak et al. determined a detailed study on generalized reverse derivations. In the first part of their article, they

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examined that the prime ring commutativity under the influence of functional identities that were involving two generalized reverse derivations. In the second part, they explored the relationships between r-generalized reverse derivations and l-generalized derivations, as well as the interplay between l-generalized reverse derivations and r-generalized derivations in the context connected to a non-central square-closed Lie ideal when a ring R was a semiprime ring.
Building on these previous studies, researchers have made significant progress in understanding the conditions under which commutativity holds in various algebraic structures. Several of these results have been derived by applying appropriate mappings such as derivations, generalized derivations, and generalized reverse derivations that satisfy specific functional identities within suitable subsets of the ring R. For more on these developments, please see references [9], [10], [11] and [12].
More recently, attention has turned to the behavior of factor rings R/P, where P is a prime ideal of R. Researches have investigated how these factor rings behave under influences of derivations, generalized derivations, generalized (α, β)-derivations that satisfy certain differential identities involving in a prime ideal. Further details are found in literature, for instance the interested reader can be referred to [[13]– [23]]. In [24] and [25]. The concept of generalized reverse derivations was placed instead of generalized derivations, and the commutativity of factor rings R/P was studied under identities involving prime ideals related to the generalized reverse derivation F. This shift in perspective has led to new insights into the algebraic structure of factor rings.
The primary aim of current article is to further study in this direction. More precisely, considering that an arbitrary ring R that equipped with generalized reverse derivations F and G that associated with reverse derivations d and g, respectively. We prove that if (F, d) and (G, g) satisfies several functional identities involving within prime ideal P, then the factor ring R/P is an integral domain. In some cases, it comes out that the range of the generalized reverse derivation F or G or the range of addition or difference of two associated reverse derivations are in a prime ideal P, i.e., $F(R) \subseteq P$, $G(R) \subseteq P$, $d(R) \subseteq P$, $g(R) \subseteq P$ or $(d \pm g)(R) \subseteq P$. Moreover, some consequences as well as special cases are concluded. Examples that illustrate the necessity of the primeness assumptions stated in our theorems are provided.

2 Preliminaries

Fact 1 *[26] Let I be a nonzero ideal of any ring R, and let P be a prime ideal of R such that $P \subsetneq I$. If $aIb \subseteq P$ for all $a, b \in R$, then either $a \in P$ or $b \in P$.*

Fact 2 *Let I be a non-zero ideal of a ring R, and let P be a prime ideal of R provided that $P \subsetneq I$. If d is a reverse derivation of R such that $d(I)$ is contained in P, then $d(R) \subseteq P$.*

Fact 3 *Let I be a non-zero ideal of a ring R, and let P be a prime ideal of R provided that $P \subsetneq I$. If (F, d) is a generalized derivation of R with $d(I) \subseteq P$, that satisfies $F(I) \subseteq P$, then $F(R)$ is also contained in P.*

Lemma 1. *[27, Lemma 2.3] If P and I be two ideals of a given ring R where P be prime provided that $P \subsetneq I$, then R/P is an integral domain if $[x, y] \in P$ satisfies for every two elements $x, y \in I$.*

Alsowait et al. in [Lemma 2. , [24]] proved that the reverse derivation d mapping a ring R to a prime ideal P or the factor ring of a ring R by a prime ideal P is integral domain, if $[x, F(x)] \in P$ for all $x \in R$. Where F is a generalized reverse derivation that associated with d. Following in similar tactic with minor treatment, we prove same identity for all $x \in I$, where I is an ideal of a ring R.

Lemma 2. *Let P and I be two ideals in an arbitrary ring R such that P is a prime ideal provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation F associated with a reverse derivation d such that $[x, F(x)] \in P$ for all $x \in I$. Then, it followed that $d(R) \subseteq P$ or R/P is an integral domain.*

Following corollary is a consequence of Lemma 2 that outcomes when we restricted a generalized reverse derivation F to be a reverse derivation d.

Corollary 1. *Let P and I be two ideals in an arbitrary ring R such that P is a prime ideal provided that $P \subsetneq I$. If R equipped with a reverse derivation d, such that $[x, d(x)] \in P$ for all $x \in R$. Then, it followed that $d(R) \subseteq P$ or R/P is an integral domain.*

3 Main Result

Bouchannafa et al. in [[28], Theorem 2.5] proved that either a ring R/P is an integral domain or an associated derivation d maps a ring R to a prime ideal P, whenever the ring R that equipped with a generalized derivation F such that $F(x \circ y) - F(x) \circ y \in Z(R/P)$ for all $x, y \in R$, where P is a prime ideal of a ring R. Alsowait et al. [24]. Theorem 1] had get similar outcome when they studied the identity $F(x) \circ y - F(x \circ y) \in P$ for all $x, y \in R$, whenever the ring R that equipped with a generalized reverse derivation F that associated with reverse derivation d. Also, they studied the identity $F(x) \circ y + F(x \circ y) \in P$ for all $x, y \in R$. The similar outcome with minor different was gotten, that was an associated derivation d maps the arbitrary ring R to a prime ideal P, or the factor of a ring R by a prime ideal P is an integral domain of characteristic two.

In the context of next theorems, our objective is to achieve the parallel outcomes for expanded identities, by utilizing two generalized reverse derivations F and G, which is associated with a reverse derivation d and g, respectively. That denoted by (F, d) and (G, g).

Theorem 1. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with generalized reverse derivations (F, d) and (G, g), that satisfies the condition $F(x) \circ y \pm G(x \circ y) \in P$ for all $x, y \in I$. Then, it followed that:*
(i) R/P is an integral domain of characteristic two, or
(iii) R/P is an integral domain and $(F \pm G)(R) \subseteq P$, or
(iii) $G(R) \subseteq P$ and $F(R) \subseteq P$.

Proof. (i) The assumption is

$$F(x) \circ y + G(x \circ y) \in P, \qquad \forall\ x, y \in I. \tag{1}$$

Placing yx instead of x in Equation (1) to yield

$$(F(x) \circ y)y + (x \circ y)d(y) + x[d(y), y] + G(x \circ y)y + (x \circ y)g(y) \in P, \quad \forall\ x, y \in I. \tag{2}$$

By right multiplying of Equation (2) by y and comparing it with Equation (1), we obtain

$$(x \circ y)(d(y) + g(y)) + x[d(y), y] \in P, \quad \forall\ x, y \in I. \tag{3}$$

Placing tx instead of x in Equation (3) to yield

$$t(x \circ y)(d(y) + g(y)) - [t, y]x(d(y) + g(y)) + tx[d(y), y] \in P, \quad \forall\ x, y, t \in I.$$

By left multiplying of Equation (3) by t and comparing it with previous equation to yield

$$[t,y]I(d(y)+g(y)) \subseteq P, \quad \forall\, y,t \in I.$$

Utilizing Fact (1), the last equation implies that $[t,y] \in P$ for all $t,y \in I$, or $d(y)+g(y) \in P$ for all $y \in I$. In **First scenario**, we conclude that R/P is an integral domain by applying Lemma 1. in light of commutativity of R/P, Equation (1) can be rewritten as

$$2F(x)y + 2G(xy) \in P, \qquad \text{for all } x,y \in I. \tag{4}$$

Placing yt instead of y in previous equation and applying it, we obtain $2xIg(t) \subseteq P$ for all $x,t \in I$. By utilizing Fact 1, we deduced that $char(R/P) = 2$ or $d(t) \in P$ for all $t \in I$. Temporary, let us assume that $char(R/P) \neq 2$, hence Equation (4) simplifies to $F(x)y + G(x)y \in P$ for all $x,y \in I$. Hence $(F+G)(x)I \subseteq P$ for all $x \in I$. According the primeness of P and initial assumption that $P \neq I$ together with Fact 3, we conclude that $(F+G)(R) \subseteq P$.
In **Second scenario**, we have $(d+g)(y) \in P$ for all $y \in I$, applying it in Equation (3), we can easily conclude that $xI[d(y),y] \subseteq P$ for all $x,y \in I$. Utilizing Fact 1 with initial assumption that $P \neq P$, we deduced $[d(y),y] \in P$ for all $y \in I$. Hence R/P is an integral domain or $d(R) \subseteq P$, by applying Corollary 1. The last result immediately leads us to $g(R) \subseteq P$.
Now, we can be rewriting Equation 1 as

$$F(x)y + yF(x) + G(xy) + G(yx) \in P, \qquad \text{for all } x,y \in I.$$

By using the result $g(R) \subseteq P$, the last expression simplifies to

$$F(x)y + yF(x) + G(y)x + G(x)y \in P, \qquad \text{for all } x,y \in I. \tag{5}$$

Using substituting $x = yx$ in Equation 5, we find

$$F(x)yy + yF(x)y + G(y)yx + G(x)yy \qquad \text{for all } x,y \in I.$$

By right multiplying of Equation 5 by y and comparing with last equation, we are having $G(y)[x,y] \in P$ for all $x,y \in I$. Taking $x = xt$, we arrive to $G(y)I[t,y] \in P$ for all $t,y \in I$. Again; by utilizing Fact 1, we can determinate that either $G(I) \subseteq P$ or $[t,y] \in P$ for all $y,t \in I$. Applying Fact 3 and Lemma 1, we conclude that $G(R) \subseteq P$ or R/P is an integral domain. By using first outcome in Equation 5, we can be written

$$F(x)y + yF(x) \in P, \qquad \text{for all } x,y \in I. \tag{6}$$

Placing tx rather than x in Equation 6, we obtain $F(x)ty + yF(x)t \in P$ for all $x,y,t \in I$. By right multiplying of Equation 6 by t and comparing with last equation, we are arriving to $F(x)[t,y] \in P$ for all $x,y,t \in I$. Taking $x = mx$, we arrive to $F(x)I[t,y] \in P$ for all $x,y,t \in I$. we can determinate that either $F(I) \subseteq P$ or $[t,y] \in P$ for all $y,t \in I$. Applying Fact 3 and Lemma 1, respectively. We conclude that $F(R) \subseteq P$ or R/P is an integral domain.

(ii) For the assumption $F(x) \circ y - G(x \circ y) \in P, \qquad \forall\, x,y \in I$. It can prove in similar technique that followed in (i).

If we restricted G to be F, then we can give the following corollaries as consequences.

Corollary 2. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation (F, d), that satisfies the condition $F(x) \circ y - F(x \circ y) \in P$ for all $x, y \in I$. Then, it followed that:*
(i) R/P is an integral domain of characteristic two, or
(ii) R/P is an integral domain and $d(R) \subseteq P$, or
(iii) $F(R) \subseteq P$.

Corollary 3. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation (F, d), that satisfies the condition $F(x) \circ y + F(x \circ y) \in P$ for all $x, y \in I$. Then, it followed that:*
(i) $char(R/P) = 2$, or
(ii) R/P is an integral domain and $d(R) \subseteq P$, or
(iii) $F(R) \subseteq P$.

Other consequence of Theorem 1 is when we restricted the two generalized reverse derivations F and G to be the associated reverse derivations d and g, respectively. That we can present in following corollary.

Corollary 4. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with reverse derivations d and g, that satisfies the condition $d(x) \circ y \pm g(x \circ y) \in P$ for all $x, y \in I$. Then, it followed that $d(R) \subseteq P$ and $g(R) \subseteq P$, or R/P is an integral domain of characteristic two.*

Example 1. Consider the ring of real numbers $\mathbb{R}$ and let $R = \{ae_{12} + be_{13} + ce_{14} + be_{24} - ae_{34} \mid a, b, c \in \mathbb{R}\}$. $I = \{ce_{14}\}$ and $P = \{0\}$. Defining $(F, d), (G, g) : R \longrightarrow R$, by:

$$F(ae_{12} + be_{13} + ce_{14} + be_{24} - ae_{34}) = -ce_{14} \quad ;$$
$$d(ae_{12} + be_{13} + ce_{14} + be_{24} - ae_{34}) = -ce_{14} + be_{24} - ae_{34},$$

and

$$G(ae_{12} + be_{13} + ce_{14} + be_{24} - ae_{34}) = -2ce_{14} \quad ;$$
$$g(ae_{12} + be_{13} + ce_{14} + be_{24} - ae_{34}) = -2ce_{14} + 2be_{24} - 2ae_{34}.$$

It is evident that R is a ring, I and P are ideals of R that satisfies $P \subsetneq I$, F and G are generalized reverse derivations associated with the reverse derivations d and g, respectively. That satisfies the exploring identity in Theorems 1. However, R/P is noncommutative and its characteristic does not equal two, $F(R) \nsubseteq P$, $G(R) \nsubseteq P$, $d(R) \nsubseteq P$, $g(R) \nsubseteq P$ and $(F \pm G)(R) \nsubseteq P$. Moreover, P is not a prime ideal of R since $ce_{14}\ (ae_{12} - ae_{34}) \in P$, but neither $ce_{14} \in P$ nor $(ae_{12} - ae_{34}) \in P$, hence; P is not prime ideal of R. Therefore, the assumption that P is prime in Theorems 1 cannot be omitted.

Example 2. Consider $R = W_3 \times \mathbb{H}$, where $W_3 = \{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C}\}$, $\mathbb{C}$ is a ring of complexes and $\mathbb{H}$ is a ring of quaternions whit integers coefficients. $I = \{(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} , 2\mathbb{H})\}$ and $P = (0, 0)$. Defining $(F, d), (G, g) : R \longrightarrow R$, by:

$$F(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} , 2\mathbb{H}) = (\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , 0) \quad ; \quad d(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} , 2\mathbb{H}) = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2c & 0 & 0 \end{pmatrix} , 0),$$

and

$$G(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}, 2\mathbb{H}) = (\begin{pmatrix} 0 & 0 & 0 \\ 2a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0) \quad ; \quad g(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}, 2\mathbb{H}) = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix}, 0).$$

It is evident that R is a ring, I and P are ideals of R that satisfies $P \subsetneq I$, F and G are generalized reverse derivations associated with the reverse derivations d and g, respectively. That satisfies the exploring identity in Theorems 1. However, R/P is noncommutative and its characteristic does not equal two, $F(R) \nsubseteq P$, $G(R) \nsubseteq P$, $d(R) \nsubseteq P$, $g(R) \nsubseteq P$ and $(F \pm G)(R) \nsubseteq P$. Moreover, P is not a prime ideal of R since $(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, 0)(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, q) \in P$, but neither $(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, 0) \in P$ nor $(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, q) \in P$. Hence; P is not prime ideal of R. Therefore, the assumption that P is prime in Theorems 1 cannot be omitted.

Theorem 2. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with generalized reverse derivations (F, d) and (G, g), that satisfies the conditions*
(1) $[F(x), y] + G(x \circ y) \in P$ *for all* $x, y \in I$, *then*
(i) $(d - g)(R) \subseteq P$, *or*
(ii) R/P *is an integral domain and* $G(R) \subseteq P$, *or*
(iii) R/P *is an integral domain of characteristic two.*

(2) $[F(x), y] - G(x \circ y) \in P$ *for all* $x, y \in I$, *then*
(i) $char(R/P) = 2$, *or*
(ii) R/P *is an integral domain and* $G(R) \subseteq P$, *or*
(iii) $(d + g)(R) \subseteq P$.

Proof. (i) The assumption is

$$[F(x), y] + G(x \circ y) \in P, \qquad \forall\ x, y \in I. \tag{7}$$

Placing yx instead of x in Equation (7) to yield

$$[F(x), y]y + x[d(y), y] + [x, y]d(y) + G(x \circ y)y + (x \circ y)g(y) \in P, \qquad \forall\ x, y \in I. \tag{8}$$

By right multiplying of Equation (8) by y and comparing it with Equation (7), we obtain

$$x[d(y), y] + [x, y]d(y) + (x \circ y)g(y) \in P, \qquad \forall\ x, y \in I. \tag{9}$$

Placing tx instead of x in Equation (9) to yield

$$tx[d(y), y] + t[x, y]d(y) + [t, y]xd(y) + t(x \circ y)g(y) - [t, y]xg(y) \in P, \qquad \forall\ x, y, t \in I.$$

Left multiplication of Equation (9) by t and comparing with last equation, we conclude

$$[t, y]I(d(y) - g(y)) \subseteq P, \qquad \forall\ x, y, t \in I.$$

Utilizing Fact 1 in the last equation, we obtain that $[t, y] \in P$ for all $t, y \in I$, or $d(y) - g(y) \in P$ for all $y \in I$. **First scenario**, we conclude that R/P is an integral domain by applying Lemma 1. Using this outcome in given hypothesis that becomes

$$2G(xy) \in P, \qquad \forall\ x, y \in I. \tag{10}$$

Placing xy instead of x in Equation (10) and using it, we can arrive to $2xIg(y) \subseteq P$ for all $x, y \in I$. Utilizing Fact 1, we obtain $2x \in P$ for all $x \in I$, or $g(y) \in P$ for all $y \in I$. First case with given that $P \neq I$, forces that the characteristic of the factor ring R/P is two. Temporarily, let $char(R/P) \neq 2$, then $g(y) \in P$ valid for all $y \in I$. Utilizing it in Equation (10), that can be rewritten as $G(y)x \in P$ for all $x, y \in I$. Again, Fact 1 together with the given assumption that $P \neq I$ force that $G(y) \in P$ for all $y \in I$. Therefore; $G(R) \subseteq P$, immediately by applying Fact 3.

Second scenario, $(d-g)(y) \in P$ for all $y \in I$, implies that $(d-g)(I) \subseteq P$. Hence, by utilizing Fact 2, we conclude $(d-g)(R) \subseteq P$.

(ii) For the assumption $[F(x), y] - G(x \circ y) \in P, \qquad \forall\, x, y \in I$. It can prove in similar technique that followed in (i).

As consequences of Theorem 2, we present the following corollaries. First one if $G = F$, second outcomes when we restricted F and G to be associated d and g, respectively.

Corollary 5. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation (F, d), that satisfies the condition $[F(x), y] - F(x \circ y) \in P$ for all $x, y \in I$. Then, it followed that:*
(i) $char(R/P) = 2$, or
(ii) $F(R) \subseteq P$, or
(iii) R/P is an integral domain and $F(R) \subseteq P$.

Corollary 6. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with reverse derivations d and g, that satisfies the condition*
(1) $[d(x), y] + g(x \circ y) \in P$ for all $x, y \in I$, then
(i) $(d-g)(R) \subseteq P$, or
(ii) R/P is an integral domain and $g(R) \subseteq P$, or
(iii) R/P is an integral domain of characteristic two.

(2) $[d(x), y] - g(x \circ y) \in P$ for all $x, y \in I$, then
(i) $char(R/P) = 2$, or
(ii) R/P is an integral domain and $g(R) \subseteq P$, or
(iii) $(d+g)(R) \subseteq P$.

Example 3. In Examples 1 and 2, we can note that the identity in Theorem 2 satisfied, although; R/P is not an integral domain and $G(R) \nsubseteq P$, $char(R/P) \neq 2$ and $(d \mp g)(R) \nsubseteq P$. This emphasize the necessity of the primeness condition in Theorem 2.

Theorem 3. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with generalized reverse derivations (F, d) and (G, g), that satisfies the condition $[F(x), y] \pm G[x, y] \in P$ for all $x, y \in I$. Then, it followed that:*
(i) $F(R) \subseteq P$ and $G(R) \subseteq P$, or
(ii) R/P is an integral domain.

Proof. (i) The assumption is

$$[F(x), y] + G[x, y] \in P, \qquad \forall\, x, y \in I. \tag{11}$$

Placing yx instead of x in Equation (11), we obtain

$$[F(x), y]y + x[d(y), y] + [x, y]d(y) + G[x, y]y + [x, y]g(y) \in P, \qquad \forall\, x, y \in I. \tag{12}$$

By right multiplying of Equation (12) by y and comparing it with Equation (11), we obtain

$$x[d(y), y] + [x, y](d(y) + g(y)) \in P, \qquad \forall\ x, y \in P. \tag{13}$$

Placing tx instead of x in Equation (13), we obtain

$$tx[d(y), y] + t[x, y](d(y) + g(y)) + [t, y]x(d(y) + g(y)) \in P, \quad \forall\ x, y, t \in I.$$

By left multiplying of Equation (13) by t and comparing it with previous equation, we are concluding

$$[t, y]I(d(y) + g(y)) \subseteq P, \qquad \forall\ t, y \in I.$$

By utilizing Fact 1, the last equation implies that $[t, y] \in P$ for all $t, y \in I$, or $d(y)+g(y) \in P$ for all $y \in I$. From first scenario, we conclude that R/P is an integral domain by applying Lemma 1. Second one; $(d+g)(y) \in P$ for all $y \in I$, implies that $(d+g)(I) \subseteq P$. Hence, by utilizing Fact 2, we conclude $(d + g)(R) \subseteq P$. Applying this outcome in Equation 13, we can deduce that $xI[d(y), y] \in P$ for all $y \in I$. By using Fact 1 together initial assumption that $I \neq P$, we determine that $[d(y), y] \in P$ for all $y \in I$. Corollary 1 is arrived us to R/P is an integral domain or $d(R) \subseteq P$. Last outcome leads to $g(R) \subseteq P$. Utilizing this result in Equation 11, we can be rewriting

$$F(x)y - yF(x) + G(y)x - G(x)y \in P, \qquad \forall\ x, y \in I. \tag{14}$$

Placing yx rather than x in Equation 14, we have $F(x)yy-yF(x)y+G(y)yx-G(x)yy \in P$ for all $x, y \in I$. Right multiplying of Equation 14 by y and comparing with last relation, we can arrive to $G(y)I[t, y] \in P$ for all $y, t \in I$. By applying Fact 1, we conclude that $G(I) \in P$ or $[t, y] \in P$ for all $y, t \in I$. Utilizing Fact 3 and Lemma 1, respectively. We conclude that $G(R) \subseteq P$ or R/P is an integral domain. Finally, using first outcome in Equation 14, we obtain

$$F(x)y - yF(x) \in P, \qquad \forall\ x, y \in I. \tag{15}$$

Using substituting $x = tx$ in Equation 15 and applying it, we obtain $F(x)[t, y] \in P$ for all $x, y, t \in I$. Letting $x = mx$, we can deduced that $F(x)I[t, y] \in P$ for all $x, y, t \in I$. Again, by applying Fact 1, we conclude that $F(I) \in P$ or $[t, y] \in P$ for all $y, t \in I$. utilizing Fact 3 and Lemma 1, respectively. We conclude that $F(R) \subseteq P$ or R/P is an integral domain.

(ii) For the assumption $[F(x), y] - G[x, y] \in P, \qquad \forall\ x, y \in I$. It can prove in similar technique that followed in (i).

As previously; if we restricted G to be F in Theorem 3, then we can give the following corollaries as consequences.

Corollary 7. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation (F, d), that satisfies the condition $[F(x), y] - F[x, y] \in P$ for all $x, y \in I$, then either $F(R) \subseteq P$ or R/P is an integral domain.*

Corollary 8. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with a generalized reverse derivation (F, d), that satisfies the condition $[F(x), y] + F[x, y] \in P$ for all $x, y \in I$, then*
(i) $char(R/P) = 2$, or
(ii) $F(R) \subseteq P$, or

(iii) R/P is an integral domain.

Also, other consequence of Theorem 3 is when we restricted the two generalized reverse derivations F and G to be the associated reverse derivations d and g, respectively. That we can present in following corollary.

Corollary 9. *Consider I and P are two ideals in any ring R, where P is prime provided that $P \subsetneq I$. If R equipped with reverse derivations d and g, that satisfies the condition $[d(x), y] \pm g[x, y] \in P$ for all $x, y \in I$. Then, it followed that:*
(i) $d(R) \subseteq P$ and $g(R) \subseteq P$, or (ii) R/P is an integral domain.

Example 4. In Examples 1 and 2, we can note that the identity in Theorem 3 satisfied, although; R/P is not an integral domain, $F(R) \nsubseteq P$ and $G(R) \nsubseteq P$. This emphasize the necessity of the primeness condition in Theorem 3.

4 Conclusion

In current work, we went ahead studied of generalized reverse derivation related to prime ideal, when we used the arbitrary assumption for a study ring R and the domain of taken elements is an ideal of R. Where R equipped with two generalized reverse derivations F and G associated with reverse derivation d and g, respectively. We proved that if (F, d) and (G, g) satisfies several functional identities involving within prime ideal P, then the factor ring R/P is an integral domain. In some cases, it came out that the range of the generalized reverse derivation F or G or the range of addition or difference of two associated reverse derivations are in a prime ideal P, i.e., $F(R) \subseteq P$, $G(R) \subseteq P$, $d(R) \subseteq P$, $g(R) \subseteq P$ or $(d \pm g)(R) \subseteq P$. Moreover, some consequences as well as special cases were concluded. Examples that illustrated the necessity of the primeness assumptions stated in our theorems are provided.

References

1. R.M. Al-omary and S. K Nauman, Generalized derivations on prime rings satisfying certain identities, Commun. Korean Math. Soc., 36(2) (2021), 229—238. https://doi.org/10.4134/CKMS.c200227
2. A.Y. Hummdi, Z. Al-Amery and R.M. Al-omery, Factor rings with algebraic identities via generalized derivations, Axioms, 14(1) (2025), 397-399. https://doi.org/10.3390/axioms14010015.
3. I.N. Herstein, N. Jordan derivations of prime rings, Proc. Am. Math. Soc. , 8 (1957), 1104–1110.
4. M.S. Samma and N. Alyamani, Derivations and reverse derivations in semiprime rings, Int. Math. Forum, 2 (2007), 1895– 1902.
5. A. Aboubakr and S. Gonzalez, Generalized reverse derivations on semiprime rings, Sib. Math. J., 56 (2015), 199–205. https://doi.org/10.1134/S0037446615020019
6. R.M. Al-omary, Commutativity of prime ring with generalized $(\alpha, \beta)-$reverse derivations satisfing certain identities, Bull. Transilv. Univ. Bras. III Math. Comput. Sci., 2 (2022), 1–12. https://doi.org/10.31926/but.mif.2022.2.64.2.1
7. A.M. Ibraheem, Right ideals and generalized reverse derivations on prime rings, Am. J. Comput. Appl. Math., 6 (2016), 162–164. https://doi.org/10.5923/j.ajcam.20160604.02
8. T. Bulak, A. Ayran and N. Aydın, Generalized reverse derivations on prime and semiprime rings, Int. J. Open Probl. Compt. Math., 14 (2021), 50–60.
9. E. Albas, Generalized derivations on ideals of prime rings, Miskolc Math. Notes, 14 (1) (2013), 3–9. https://doi.org/10.18514/MMN.2013.499
10. Ö.Ayat, N. Aydin and B. AlBayrak, Generalized reverse derivation on closed Lie ideals, J. Sci. Perspect., 2 (2018), 61–74. https://doi.org/10.26900/jsp.2018342245
11. M. Dadhwal and Neelam, On derivations and Lie structure of semirings, Malaya J. Mat., 12 (2) (2024), 206–217. https://doi.org/10.26637/mjm1202/006
12. S. Huang, Generalized reverse derivations and commutativity of prime rings, Commun. Math., 27 (2019), 43–50. https://doi.org/10.2478/cm-2019-0004.hal03664963

13. F. Almahdi, A. Mamouni and M. Tamekkante, A generalization of Posner's theorem on derivations in ring, Indian J. Pure Appl. Math., 51 (2020), 187–194. https://doi.org/10. https://doi.org/10.1007/s13226-020-0394-8
14. N. Alsowait, R.M. Al-omary, Z. Al-Amery and M. Al-Shomrani, Exploring commutativity via generalized (α, β)-derivations involving prime ideals, Mathematics, 12 (2024), 2325. https://doi.org/10.3390/math12152325
15. N.L. Alsowait, M. Al-Shomrani, R.M. Al-omary, Z.Z. Al-Amery and H. Alnoghashi, On ideals and behavior of quotient rings via generalized (α, β)-derivations, Mathematics., 13 (6) (2025), 968. https://doi.org/10.3390/math13060968
16. A. Boua and G.S. Sandhu, Results on Various Derivations and Posner's Theorem in Prime Ideals of Rings, Bol. Soc. Paran. Mat., 3 (2023), 1-13. https://doi.org/10.5269/bspm.62478
17. A.Y. Hummdi, R.M. Al-omary and Z.Z. Al-Amery, Some notes on generalized P-derivations with ideals in factor rings, Contemp. Math., 6 (2025), 2148–2158. https://doi.org/10.37256/cm.6220256460
18. A. Mamouni, L. Oukhtite and M. Zerra, On derivations involving prime ideals and commutativity in rings, Sao Paulo J. Math. Sci., 14 (2020), 675–688. https://doi.org/10.1007/s40863-020-00187-z
19. A. Mamoun, L. Oukhtite and M. Zerra, Some results on derivations and generalized derivations in rings, Mathmatica, 65 (2023), 94–104. https://doi.org/10.24193/mathcluj.2023.1.10
20. H.E. Mir, A. Mamouni and L. Oukhtite, Commutativity with algebraic identities involving prime ideals, Commu. Korean Math. Soc., 35 (2020), 723–731. https://doi.org/10.4134/CKMS.c190338
21. N. Rehman, H. Alnoghashi, and M. Honagn, On generalized derivations involving prime ideals with involution, Ukr. Math. J., 75 (2024), 1219–1241. https://doi.org/10.1007/s11253-023-02257-9
22. N. Rehman, N. Alnoghashi and A. Boua, Identities in a prime ideal of a ring involving generalized derivations, *Kyungpook Math. J.*, 61 (2021), 727–735. https://doi.org/10.5666/KMJ.2021.61.4.727
23. M. Zerra, K. Bouchannafa and L. Oukhtite, On generalized derivations in factor rings, Georg. Math. J., 32(1) (2025), 175–182. https://doi.org/10.1515/gmj-2024-2017
24. N. Alsowait, M. Al-Shomrani, R.M. Al-omary and Z.Z. Al-Amery, On a quotient ring that satisfies certain identities via generalized reverse derivations, Mathematics, 13(5) (2025), 870. https://doi.org/10.3390/math13050870
25. A.K. Faraj and A.M. Abduldaim, Commutativity and prime ideals with proposed algebraic identities, Int. J. Math. Comput. Sci., 16 (2021), 1607–1622.
26. L. Oukhtite and K. Bouchannafa, Commutativity of rings with constraints on endomorphism, Rend. Circ. Mat. Palermo, II. Ser. 72 (2023), 157-1581. https://doi.org/10.1007/s12215-022-00754-7
27. N. Rehman, E. K. S*ögütcü* and H. M. Alnoghashi, A generalization of Posner's theorem on generalized derivations in rings, J. Iranian Math. Soc., 3(1) (2022), 1–9. https://doi.org/10.30504/JIMS.2022.335190.1059
28. K. Bouchannafa; M.A. Idrissi; L. Oukhtite, Relationship between the structure of a quotient ring and the behavior of certain additive mapping, Commun. Korean Math. Soc., 37 (2022), 359–370. https://doi.org/10.4134/CKMS.c210126

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