MULTIPOINT MOVING NODES FOR P ARABOLIC EQUATIONS

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ABSTRACT

This word discusses approximate solutions of linear parabolic equations with initial-boundary conditions. The primary focus is on methods that effectively find such solutions by employing a moving finite difference analog of the differential equation. This approach allows us to formulate an approximate analytical solution, significantly simplifying the computation process. By transitioning from the differential equation to an algebraic equation, we obtain a single equation, the solution of which represents an approximate analytical solution to the original problem. However, to achieve higher accuracy in this solution, we apply additional moving nodes, which enhances the results. By using multipoint moving nodes, we can form a system of algebraic equations, the solution of which provides us with an improved analytical solution. The article also presents numerical experiments that confirm the effectiveness of the proposed method and its advantages over traditional approaches.

KEYWORDS

Boundary conditions, differential equation, multipoint moving nodes, initial-value problem.

1. INTRODUCTION

This article discusses approximate solutions of linear parabolic equations with initial-boundary conditions. The primary focus is on methods that effectively find such solutions by employing a moving finite difference analog of the differential equation. This approach allows us to formulate an approximate analytical solution, significantly simplifying the computation process.

By transitioning from the differential equation to an algebraic equation, we obtain a single equation, the solution of which represents an approximate analytical solution to the original problem. However, to achieve higher accuracy in this solution, we apply additional moving nodes, which enhances the results.

By using multipoint moving nodes, we can form a system of algebraic equations, the solution of which provides us with an improved analytical solution. The article also presents numerical experiments that confirm the effectiveness of the proposed method and its advantages over traditional approaches.

Many phenomena and processes in nature are described using differential equations. These equations are a powerful tool for modeling various physical, chemical, and engineering processes. For example, one of the most common applications is heat conduction in solid bodies. In this case, differential equations help describe how temperature is distributed within a material over time, taking into account factors such as thermal conductivity and initial conditions.

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Additionally, mass transfer processes in chemical technology can also be modeled using differential equations. Here, it is important to consider how different components interact with each other and how they move through space, which requires solving initial-boundary value problems of parabolic type.

Another example is the movement of liquids, where it is necessary to account for the influence of temperature on the dynamics of flow. In such cases, differential equations allow us to describe how temperature affects the viscosity of the liquid and, consequently, its motion.

Thus, initial-boundary value problems of parabolic type play a crucial role in modeling and analyzing these complex processes, providing a deeper understanding of the phenomena occurring in nature[1-3].

Analytical methods do not possess the necessary universality for solving such problems. While analytical approaches can be effective in certain cases, they often face limitations when dealing with complex systems or equations for which it is impossible to find an exact solution.

In such situations, alternative methods become more universal. For example, projection methods allow for transforming problems into more manageable forms, facilitating their solution. These methods are often used in numerical calculations and approximate solutions.

Variational methods also represent a powerful tool based on the search for extrema of functionals. They find wide application in various fields, including physics and optimization, and enable the finding of approximate solutions for complex problems.

Another important approach is the method of small parameters, which is used to simplify equations when one of the parameters of the system is significantly smaller than the others. This allows for obtaining simpler and more manageable solutions while preserving the essential characteristics of the system.

Operational methods provide another level of universality, allowing for the solution of differential equations through transformations such as the Laplace transform. These methods are particularly useful for analyzing systems in the frequency domain.

Finally, various iterative methods offer the possibility of approximating solutions to complex problems, starting from simple initial assumptions and gradually improving them. These methods are frequently used in numerical calculations and modeling, enabling the finding of solutions when analytical approaches are ineffective.

Thus, the use of more universal methods, such as projection, variational, small parameter methods, operational, and iterative methods, significantly expands the capabilities for solving complex problems and allows for obtaining more accurate and reliable results[4,5,6,7].

he method of moving nodes is classified as a universal method. In work [8], an analytical expression for the solution of two-point boundary value problems of convection-diffusion with constant coefficients was obtained using multi-point moving difference schemes.

Work [9] discusses the application of the moving nodes method for solving two-point boundary value problems based on the control volume method [10], which is known in Russian literature as the integral-interpolatory method [11].

Thus, the method of moving nodes demonstrates its universality and effectiveness in various fields, making it a valuable tool for solving complex problems.

The application of the moving nodes method in its one-point variant to partial differential equations is discussed in work [12].

The application of the moving nodes method for parabolic equations in various aspects of the one-point variant is presented in work [13]. It demonstrates the possibility of improving the solution through partial approximation of differential operators based on the direct method approach. The goal of this research is to develop a multi-point variant of the moving nodes method for parabolic-type equations.

2. Statement of the Problem

Let's consider the boundary value problem

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

with initial

$$u(x,0) = \varphi(x) \tag{2}$$

and boundary conditions

$$u(0,t) = u_0(t),\ u(b,t) = u_b(t). \tag{3}$$

In (1) – (3) $u(x,t)$ isthe unknown function, $f(x,t), \varphi(x), u_0(t)$ and $u_b(t)$ the given functions, Equation (1) is considered in the region $\Omega = 0 \le x \le b,\ 0 \le t \le T,\ \ (\sigma > 0),$

There are various methods for solving the problem: analytical and numerical. It is assumed that the initial and boundary conditions are such that the solution to the problem exists and is unique. To solve the problem, an approximate method is proposed here based on a multi-point version of the moving node method, which allows obtaining a refined approximate solution to the problem.

3. Solution Method and Discussion

As already mentioned above, dedicated to the solution of a parabolic equation, a single movable node is used. To improve the solution in the case of two-point boundary value problems, multi-point movable nodes were used in works [8,9].

The use of multi-point movable nodes for a parabolic equation faces certain difficulties associated with the loudness of mathematical expressions. In this regard, we use the capabilities of symbolic mathematics packages.

In the case of two-point boundary value problems [8,9], movable nodes were used: single-point $\omega_1 = \{0,\ x,\ b\},$, three-point $\omega_3 = \{0,\ x/2, x, (b+x)/2,\ b\}$, etc. In the given templates, nodes 0 and b coincide with the boundary point of the problem, and x is a movable node. Index in ω means the number of movable nodes.

In the work [13] in Ω one movable node is used $\omega_{11} = \omega_{x1} \times \omega_{t1}$, where $\omega_{x1} = \{0, x, b\}$, $\omega_{t3} = \{0, t/3, 2t/3, t\}$. Now the index in ω denotes the number of movable nodes by variable, for example, ω_{x3} indicates that three movable nodes are used by variable x. We introduce additional movable nodes:

$$\omega_{x3} = \{0, x/2,\ x,\ (b+x)/2,\ b\}; \quad \omega_{t2} = \{0, t/2,\ t\}; \quad \omega_{t3} = \{0, t/3,\ 2t/3,\ t\}.$$

Let us first consider the approximation of equation (1) with one movable node, i.e. in ω_{11} with an implicit scheme:

$$\frac{U(x,t)-U(x,0)}{t} = \sigma \frac{b}{2}\left[\frac{U(b,t)-U(x,t)}{b-x} - \frac{U(x,t)-U(0,t)}{x}\right] + f(x,t) \qquad (4)$$

In the equation U(x,t) is an approximate value of the unknown function at the points (x,t). Using the boundary and initial conditions, we have,

$$U(x,0) = u(x,0) = \varphi(x),\ U(0,t) = u(0,t) = u_0(t),\ U(b,t) = u(b,t) = u_b(t)$$

Solving equations (4) taking into account the boundary and initial conditions, we obtain

$$U(x,t) = \frac{x(b-x)}{2\sigma t + x(b-x)}\varphi(x) + \frac{2\sigma t[u_b(t)x + u_0(t)(b-x)]}{b(2\sigma t + x(b-x))} + \frac{x(b-x)tf(x,t)}{x(b-x)+2\sigma t} \qquad (5)$$

(5) is an approximate analytical solution to problem (1)-(3).

The approximate solution satisfies conditions (2) and (3).

To improve the solution, we add movable nodes. When approximating by template, we obtain three types of movable nodes; (x/2,t), (x,t) and ((b+x)/2,t). For each movable node, we use three implicit schemes of type (4). From this system of equations, we determine the solution, which has the form;

$$U(x,t) = \left\{\frac{M\varphi(x)}{2\sigma t + M} + \frac{4\sigma t M}{b(2\sigma t + M)}\left[\frac{(b-x)\varphi((b-x)/2)}{2M_2} + \frac{(x-a)}{2M_1}\right] + \frac{2(2\sigma t)^2 M}{b(2\sigma t + M)}\left[\frac{u_b(t)}{(b-x)(2\sigma t + M_2)} + \frac{u_0(t)}{x(2\sigma t + M_1)}\right]\right\} : \left\{1 - \frac{2(2\sigma t)^2 M}{(b-a)(2\sigma t + M)}\left[\frac{1}{(b-x)M_2} + \frac{1}{xM_1}\right]\right\} \qquad (6)$$

In (6), the notations are introduced

$$M = x(b-x)/4, \quad M_1 = 2\sigma t + (x-a)^2/2, \quad M_2 = 2\sigma t + (b-x)^2/2. \quad ,$$

Formula (6) is derived for $f(x,t)=0$.

Now, let us consider improving the solution by adding alternating nodes with respect to the variable t. For brevity, we introduce the notations;

$$P(t)=2\sigma t+x(b-x),\quad S_1(t)=\frac{x(b-x)}{P(t)},\quad S_2(t)=\frac{2\sigma t[u_b(t)x+u_0(t)(b-x)]}{P(t)},$$
$$S_3=S_1(t)t\cdot f(x,t).$$

Based on these notations, (5) can be written as follows

$$U(x,t)=S_1(t)\varphi(x)+S_2(t)+S_3(t). \tag{7}$$

Note that the variable x is not explicitly specified in the expressions $P(t),\ S_1(t),\ S_2(t),\ S_3(t)$; there is no need for this.

Let us consider the pattern $\varpi_{12}=\varpi_{x1}\times\varpi_{t2}$, i.e. movable nodes; (x,t/2) and (x,t). Let us write formula (7) for these nodes.

For node (x,t/2)

$$U(x,t/2)=S_1(t/2)\varphi(x)+S_2(t/2)+S_3(t/2).,\tag{8 }$$

and for node (x,t) ,

$$U(x,t)=S_1(t)U(x,t/2)+S_2(t)+S_3(t).\tag{9 }$$

Substituting expression (8) into (9), we obtain

$$U(x,t)=S_1(t)[S_1(t/2)\varphi(x)+S_2(t/2)+S_3(t/2)]+S_2(t)+S_3(t).\tag{10}$$

If we use three-point moving nodes by variable t, we have

$$U(x,t)=S_1(t)\{S_1(2t/3)[S_1(t/3)\varphi(x)+S_2(t/3)+S_3(t/3)]+S_2(2t/3)+S_3(2t/3)\}+ S_2(t)+S_3(t).\tag{11}$$

We can increase the number of moving nodes by both variables. Due to the difficulty of implementing this procedure, we will present this routine work to symbolic mathematics packages, in particular, we will use Maple tools.

Note that for convenience, the approach was presented for a simple parabolic equation (1). The sequence of presentation and the method for obtaining an approximate solution do not change if, instead of equation (1), we consider a more general form.

4. NUMERICAL EXPERIMENTS

We will study several examples in which the exact solution is known.

Problem 1. Consider problem (4.1)-(4.3) with the following input parameters: $\sigma = 1, \quad b = 1,$ $T = 1, \quad f(x,t) = exp(-x)(1-t) + x, \quad \varphi(x) - 0, \;\; u_0(t) = t, \;\; u_b(t) = t/e + t.$
Exact solution to the problem

$$u(x,t) = exp(-x)t + tx.$$

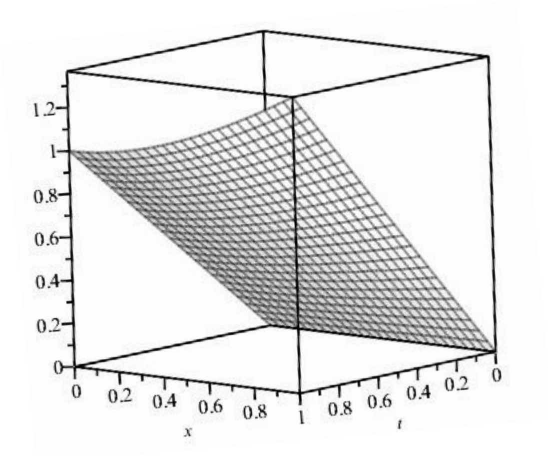

Fig.1. Exact solution for problem 1

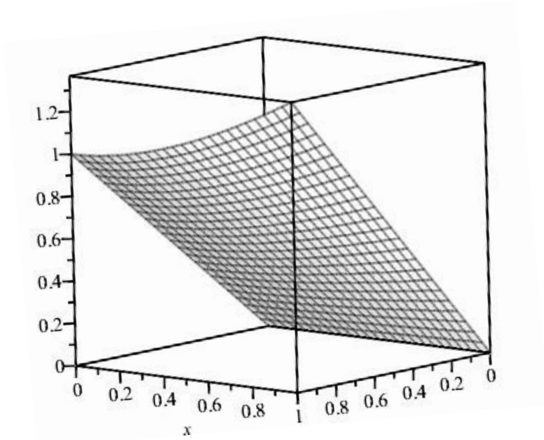

Fig. 2 Approximate solution problem 1

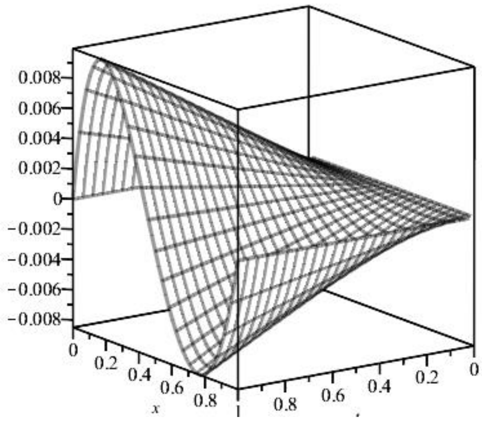

Fig.3 Difference solution; u(x,t)-U(x,t) with one movable node

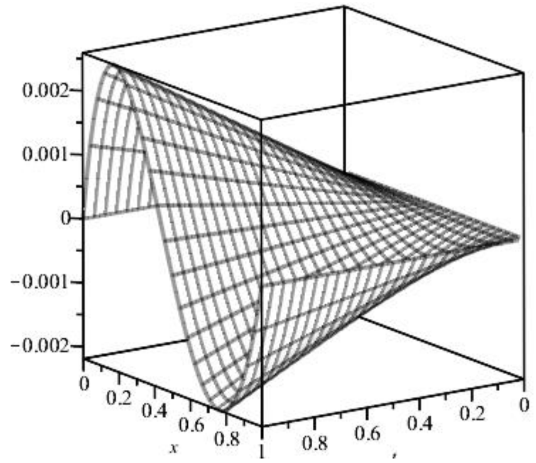

Fig.4 Difference solution; u(x,t)-U(x,t) with three movable nodes along x

Fig. 1, 2 shows a comparison of the exact and approximate solutions obtained using one movable node. It is very difficult to distinguish visually, so Fig.3 shows the difference between the exact and approximate solutions, showing a good result. When used with three movable nodes along the variable x, the result improves (the difference between the solution is shown in Fig. 4

Problem 2. Let the parameters of problem (1)-(3) have the form: $\sigma = 1, \quad b = \pi, \; T = 1,$
$T = 1, \quad f(x,t) = 0, \quad \varphi(x) = \sin x + x/\pi, \;\; u_0(t) = 0, \; u_b(t) = 1.$

Exact solution of the problem

$$u(x,t) = exp(-t)\sin(x) + x/\pi.$$

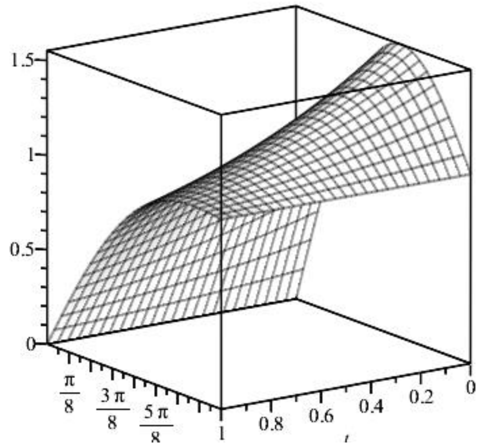

Fig.5. Exact solution problem 2

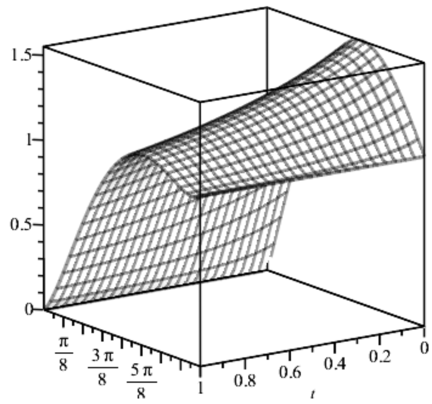

Fig.6. Approximate solution with one movable node for problem 2

Fig. 5-8 reflects a comparative analysis of the approximate and exact solutions for problem 2. The exact solution of problem 2 is shown in Fig. 5 and the approximate solution with one movable node is shown in Fig. 6. Comparison of Fig5 and Fig. 6 shows qualitative agreement between the obtained solutions. The difference between the exact and approximate solutions with a single-point moving node is shown in Fig. 7. To refine the approximate solution, additional alternating nodes were used. When using three moving nodes along the x variable and along the t variable, the difference between the exact and approximate solutions was reduced by two times. compared to one moving node (Fig. 8).

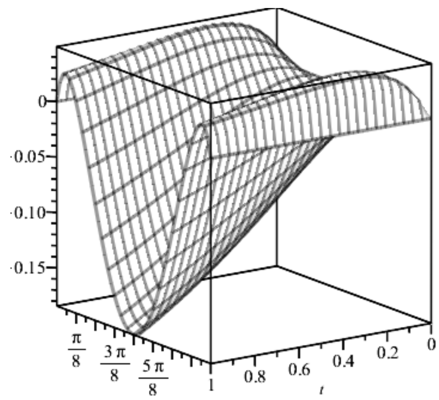

Fig.7 Difference solution; u(x,t)-U(x,t) with one movable node for problem 2

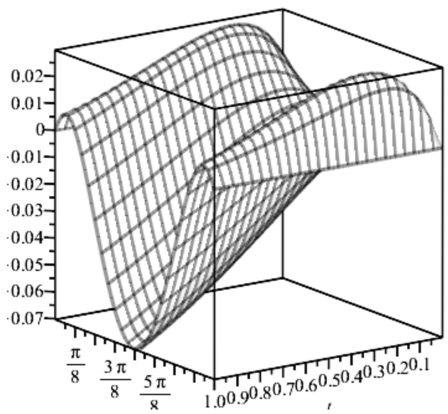

Fig.8. Difference solution; u(x,t)-U(x,t) with three movable nodes along x and along t for problem 2

Problem 3. Now consider problem (1)-(3) with the input parameters: $\sigma = 1, \quad b = 1, \; T = 1,$
$f(x,t) = 0, \quad \varphi(x) = \sin(\pi x), \; u_0(t) = 0, \; u_b(t) = 0.$

Exact solution to the problem

$$u(x,t) = exp(-\pi^2 t)\sin(\pi x).$$

The solution to this problem with one moving node is;

$$U(x,t) = \frac{x(1-x)\sin(\pi x)}{2t + x(1-x)}.$$

In this problem, one can also find a qualitative coincidence between the exact and approximate solutions with one moving node. To refine it, we increase the number of moving nodes.

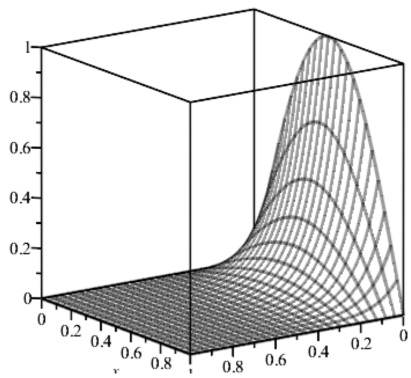

Fig.9 Exact solution for problem 3

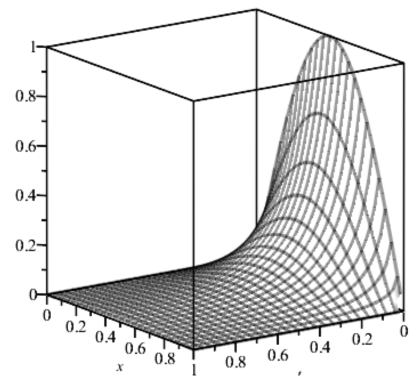

Fig.10. Approximate solution with three movable nodes in x and t for problem 3

Figures 9 and 10 show the exact and approximate solutions to problem 3. The approximate solution was obtained with three moving nodes in both variables, which shows good agreement between the approximate solution and the exact solution.

Fig. 11-13 shows graphs showing errors (the difference between the exact and approximate solutions) depending on the number of movable nodes.

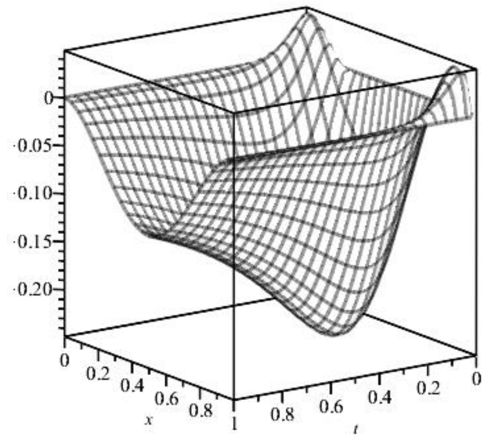

Fig.11. Error at ϖ_{11} for problem 3.

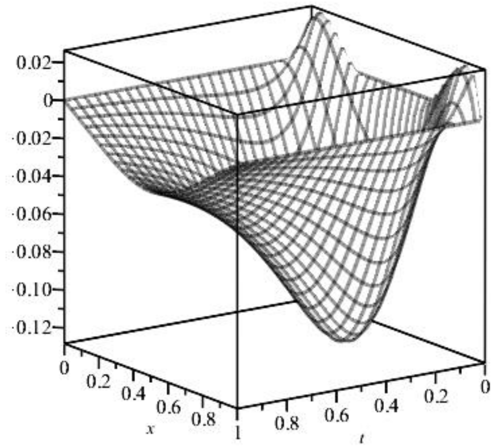

Fig.12. Error at ϖ_{32} for problem 3.

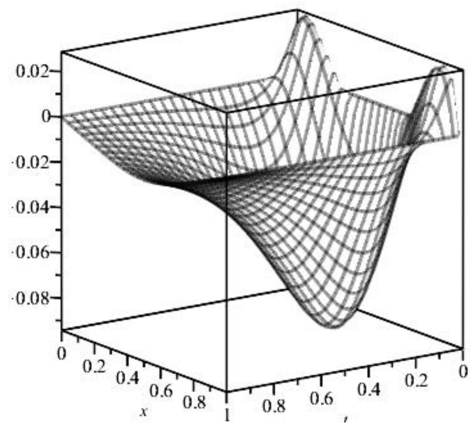

Fig.13. Error at ϖ_{33} for problem 3.

Increasing the number of movable nodes will lead to a decrease in the error of the solutions obtained. In this case, the decrease in error, in this problem, is more noticeable with an increase in the number of movable nodes in the variable t.

5. CONCLUSIONS

The presented analysis of the initial boundary value problems for parabolic equations shows that the proposed solution method allows for its application in finding approximate solutions. This method, based on moving nodes, enables adaptation to changing conditions of the problem and improves the accuracy of calculations. Increasing the number of moving nodes contributes to a more detailed approximation of the solution, which, in turn, leads to the refinement of the obtained results. Thus, the use of a greater number of nodes not only enhances accuracy but also expands the applicability of this method in various scientific and engineering tasks. Note that in this paper, the method of multipoint moving nodes is applied to a simple parabolic equation. Given the versatility of multipoint moving nodes, this solution technique can also be effectively applied to parabolic equations that include convection-reaction terms and Neumann-type boundary conditions

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