THE WEAK SOLUTION OF BLACK-SCHOLE'S OPTION PRICING MODEL WITH TRANSACTION COST

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ABSTRACT

This paper considers the equation of the type
$-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\mathfrak{P}\frac{\partial u}{\partial x}=\bar{k}F\left(\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right)(x,t)$ □ $\mathbb{R}\times(0,T)$;
which is the Black-Scholes option pricing model that includes the presence of transaction cost. The existence, uniqueness and continuous dependence of the weak solution of the Black-Scholes model with transaction cost are established.The continuity of weak solution of the parameters was discussed and similar solution as in literature obtained.

KEYWORDS

Black-Scholes Model, Option pricing, Transaction costs, Weak solution,Sobolev space

1. INTRODUCTION

Options are financial instrument that convey the right but not the obligation to engage in a future transaction on the underlying assets.In a complete financial market without transaction costs, the celebrated Black-Sholes no-arbitrage argument provide not only a rational option pricing formula but also a hedging portfolio that replicate the contingent claim [8] .However the Black-Scholes hedging portfoliorequires trading at all-time instants and the total turnover of stock in the time interval $[0,T]$ is infinite. Accordingly, when transaction cost directly proportional to trading is incorporated in the Black –Scholes model the resulting hedging portfolio is quite expensive. The condition under which hedging can take place has to be relaxed such that the portfolio only dominate rather than replicate the value of the European call option at maturity .The first model in that direction was presented in [4] . Here it was assumed that the portfolio is rebalanced at a discrete time and that the transaction cost in buying and selling the asset are proportional to the monetary value of the transaction. At a price S and a constant K depending on an individual's aversion to risk, the transaction costs are$N\,/\,K\,/\,S\,/$, where N is the number of shares bought $(N>0)$ or sold $(N<0)$. In [7], the existence, uniqueness and continuity of the Black –Scholes model was discussed. Also in [6], option pricing with transaction costs that leads to a nonlinear equation was investigated.In a related paper [1],the discretetime, dominating policies was presented. In [3] further work on this in the presence of transaction cost was presented...By applying the theorem of central limit, they show that as the time step Δt and transaction cost □ tend to zero. The price of discrete option converged to a Black –Sholes price with adjusted volatility$\hat{\sigma}\,(.)$. Here Δtrepresent the mean time length for a change in the value of the stock instead of transaction frequency. Here our adjusted volatility is given by;

$$\hat{\sigma}^2=\sigma^2(1-Lesgn(\partial_s^2 v). \qquad (1.1)$$

If

$$\hat{\sigma} = \hat{\sigma}(\partial_s^2 f), \quad Le = -\left(\sqrt{\frac{2}{\pi}} \frac{\square}{\sigma\sqrt{dt}} \right)$$

then we have the adjusted volatility as in[7], where dt is the time lag between consecutive transaction.

Let $f(s,t)$ be the value of the option and M be the value of the hedge portfolio. We assumeinstead that the value of the underlying follows the random work

$$ds = \bar{u} s dt + \hat{\sigma} s \square dw$$

with□ drowned from a normal distribution,$\bar{u}$ is a measure of the average rate of growth of the asset price also known as the drift where$\bar{u} = \alpha - r$ and σ is a measure of the diffusion coefficient . Then the change in the value of the portfolio over the time step dt is given by (using (1.1))

$$\Delta M = \sigma s \left(\frac{\partial f}{\partial s} - \Delta \right) \phi dw + \left(\frac{1}{2} \sigma^2 s^2 \square^2 (1 - Lesgn(\partial_s^2 f)) \frac{\partial^2 f}{\partial^2 s} + (\alpha - r) s \frac{\partial f}{\partial s \partial} + \frac{\partial f}{\partial t} - \alpha \Delta s \right) dt - S \, |N| K.$$

Making a digression and investigating the nature of the number of assets bought or sold given that we posit the number of asset x (the delta of our option) as$x = \frac{\partial f}{\partial s}$.

Conventionally, thedelta of an option is represented by Δ . Given that x is evaluated at the asset value s and time t,we have $x = \frac{\partial f}{\partial s}(s,t)$.

Rehedging after finite time Δt leads to a change in the value of assets as below

$$x = \frac{\partial f}{\partial s}(s + \Delta s, t + \Delta t).$$

This of course evaluated at the new asset price and time .therefore the number of assets to be traded after Δt is given by

$$v = \frac{\partial f}{\partial s}(s + \Delta s, t + \Delta t) - \frac{\partial f}{\partial s}(s,t) \approx \frac{\partial^2 f}{\partial^2 s} \sigma \square s dw.$$

Hence the expected transaction cost over a time step is

$$E[s \, |n| k] = ksE[|n|] = ksE\left[\left| \frac{\partial^2 f}{\partial^2 s} \sigma \square s dw \right| \right]$$

$$= k\sigma s^2 \left| \frac{\partial^2 f}{\partial^2 s} \right| E[|\square|] dw,$$

Where

$$|\square| = \sqrt{\frac{2}{\pi}}.$$

The expected change in the value of the portfolio is

$$E(\Delta M) = (\frac{\partial f}{\partial t} + \frac{1}{2}\hat{\sigma}^2(\hat{\sigma}_s^2 f, s, t)s^2\frac{\partial^2 f}{\partial^2 s} - k\sigma s^2\sqrt{\frac{2}{\pi dt}\left|\frac{\partial^2 f}{\partial^2 s}\right|})dt$$

If the holder of the option expects to make as much from his portfolio as from a bank account at a riskless interest rate r(no arbitrage), then

$$E(\Delta M) = r\left(f - s\frac{\partial f}{\partial s}\right)dt.$$

Hence following[9] for option pricing with transaction costs is given by

$$\frac{\partial f}{\partial t} + \frac{1}{2}\hat{\sigma}^2(\partial^2 f, s, t)s^2\frac{\partial^2 f}{\partial^2 s} + rs\frac{\partial f}{\partial s} - rf - k\sigma s^2\sqrt{\frac{2}{\pi dt}\left|\frac{\partial^2 f}{\partial^2 s}\right|} = 0, (s,t) \in (0,\infty) \times (0,T), \quad (1.2)$$

and the final condition

$$f(s,t) = \max(s - E, 0), s \in (0,\infty),$$

for European call option with strike price E.

Note that equation (1.2) contents the usual Black-Scholes terms with additional nonlinear term modeling the presence of transaction costs. Setting

$$x = \log(S/_E), t = T - \tau/\frac{1}{2}\hat{\sigma}^2, f = EV(X,\tau),$$

equation (1.2) becomes

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV = \bar{k}\ , \ (x,r) \in (0,T'), \quad (1.3)$$

with initial condition

$$V(x,0) = \max(e^x - 1,0), x \in \mathbb{R},$$

Where

$$k = r/(\hat{\sigma}^2/2), \bar{k} = k\sqrt{8/(\pi\hat{\sigma}^2 dt)}\ and\ T' = \hat{\sigma}^2 T/2.$$

Set

$$V = (x,\tau) = e^x U(x,\tau).$$

Then (1.3)gives

$$-\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2} + (k+1)\frac{\partial u}{\partial x} = \bar{k}\left|\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right|, \ (x,\tau) \in \mathbb{R} \times (0,T), \quad (1.4)$$

With the initial condition

$$u(x,0) = max(1 - e^{-x}, 0)$$

Let

$$\mathfrak{P} = k + 1$$

The previous discussion motivates us to consider the following problem that includes cost structures that go beyond proportional transaction costs.

$$-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\mathfrak{P}\frac{\partial u}{\partial x}=\bar{k}F\left(\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}\right)(x,t) \in \mathbb{R}\times(0,T), \quad (1.5)$$

and

$$u(x,0)=u_0(x), \quad x \in \mathbb{R}. \quad (1.6)$$

In this paper we looked into parameters that are governing the Black-Scholes option pricing model with the present of transaction costs such that equation (1.5) exhibits the desired behaviour. More precisely, let

$$\mathcal{P}_{ad}=\left\{q=\left(\mathfrak{P},\bar{k}\right)\in[\mathfrak{P}_{min},\mathfrak{P}_{max}]\times\left[\bar{k}_{min},\bar{k}_{max}\right]\right\},$$

where

$$\mathfrak{P}_{min}>0 \; and \bar{k}_{min}>0.$$

Defined a functional $J(q)$ by

$$J(q)=\|u(q,t)-z_d\|^2_{L^2(0,T;H)}, \quad (1.7)$$

where the daltaz_d can be thought of as the desired value of $u(q;t)$. The parameter identification problem for (1.5) with the objective function (1.7) is to find

$$q^{*}=\left(\mathfrak{P}^{*},\bar{k}^{*}\right)\in\mathcal{P}_{ad}$$

Satisfying

$$J(q^{*})=inf_{q\in\mathcal{P}_{ad}}J(q). \quad (1.8)$$

Let

$$q\rightarrow u(q)$$

from,$\mathcal{P}$ in to $C([0,T];H$ be the solution map . In what follows, the existence and uniqueness of the weak solution of (1.5) is established in the next section. Continuity of the solution with respect to data is established in section 3.

2. EXISTENCE AND UNIQUENESS OF WEAK SOLUTION

Since the type of equation in (1.5) do not belong to $L^2(\mathcal{R})$ we introduce weighted lebessgue and sobolev spaces

$$L^2_\beta(\mathcal{R})\text{and}H^1_\beta(\mathcal{R}) \text{ for } \beta>0$$

as follows.

$$L^2_\beta(\mathcal{R})=\left\{u\in L^1_{loc}(\mathcal{R}): ue^{-\beta|x|}\in L^2(\mathcal{R})\right\} \quad (2.1)$$

$$\mathrm{H}^1_\mathrm{B}(\mathcal{R})=\left\{u\in L^1_{loc}(\mathcal{R}): ue^{-\beta|x|}\in L^2(\mathcal{R}), u'e^{-\beta|x|}\in L^2(\mathcal{R})\right\}. \quad (2.2)$$

The respective inner products and norms are defined by

$$(u, v)_{L^2_\beta(\mathcal{R})} = \int_{\mathcal{R}} uve^{-2\beta|x|}dx \quad (2.3)$$

$$(u, v)_{H^1_\beta(\mathcal{R})} = \int_{\mathcal{R}} uve^{-2\beta|x|}dx + \int_{\mathcal{R}} u'v'e^{-2\beta|x|}dx \quad (2.4)$$

$$\|u\|_{L^2_\beta(\mathcal{R})} = \left(\int_{\mathcal{R}} |u|^2 e^{-2\beta|x|}dx\right)^{\frac{1}{2}} \quad (2.5)$$

$$\|u\|_{H^1_\beta(\mathcal{R})} = \left(\int_{\mathcal{R}} |u|^2 e^{-2\beta|x|}dx + \int_{\mathcal{R}} |u'|^2 e^{-2\beta|x|}dx\right)^{\frac{1}{2}} \quad (2.6)$$

We define the dual space of $H^1_\beta(\mathcal{R})$ as

$$\left(H^1_\beta(\mathcal{R})\right)^{*} = \{u \backslash u: H^1_\beta(\mathcal{R}) \to \mathcal{R} \text{ is linear and continuous}\} \quad (2.7)$$

The duality pairing between $H^1_\beta(\mathcal{R})$ and $\left(H^1_\beta(\mathcal{R})\right)^{*}$ is given by

$$\langle u, v\rangle = \int_{\mathcal{R}} |u|^2 e^{-2\beta|x|}dx \quad (2.8)$$

In what follows, we state,

LEMMA1: Let $f = L^2_\beta(\mathcal{R})$. For $\phi \in C_0^\infty$, $supp\,\phi = (-1,1)$, $\int_{\mathcal{R}} \phi d\,(x)dx = 1$, and $\phi_\epsilon = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right)$,

then

$$\phi_\epsilon * f \to f \text{ in } L^2_\beta(\mathcal{R}). \quad (2.9)$$

PROOF: Suppose $q = e^{-2\beta|x|}$,

then we have

$$(\phi_\epsilon * f).q = \left(\emptyset_\epsilon * (f.q) + (\phi_\epsilon * f).q - \phi_\epsilon * (f.q)\right). \quad (2.10)$$

since

$$f.g \in L^2 \text{ and } \phi_\epsilon * (f.q) \text{ in } L^2$$

it suffices to show that

$$\|g_\epsilon\|_{L^2} = (\|(\phi_\epsilon * f).q - \phi_\epsilon * (f.q)\|) \to 0 \qquad for\ \epsilon \to 0 \quad (2.11)$$

The fundamental theory of calculus for q give

$$g_\epsilon(x) = \int_{\mathcal{R}} \phi_\epsilon(x-y)f(y)(q(x) - q(y))\,dy \quad (2.12)$$

using

$$supp\,\square_\epsilon = (-\epsilon, \epsilon)$$

We get

$|g_\epsilon(x)| \le \int_{\mathcal{R}} |\square_\epsilon(x-y)||f(y)|(2\epsilon\, sup|q'(t)|)dy = \int_{\mathcal{R}} |\square_\epsilon(x-y)||f(y)|(2\epsilon\, sup|q'(y+s)|)dy = \overline{g_\epsilon}(x)$(2.13)

Since

$$\overline{g_\epsilon}(x) = L^2$$

uniformly and

$|g_\epsilon(x)| \le 2\epsilon|\overline{g_\epsilon}(x)|,$

thus

$\|g_\epsilon(x)\|_{L^2} \to 0$ as $\epsilon \to 0$

LEMMA 2: $D(\mathcal{R})$the space of test function in $\mathcal{R}$,is densein $H^1_\beta(\mathcal{R})$.

PROOF .Let $f \in H^1_\beta(\mathcal{R})$ $and\ \Phi \in C^\infty$

such that

$$\Phi(x) = \begin{cases} 1, & if|x| \le 1 \\ 0, & if|x| \ge 2 \end{cases}$$

Now we show that

$$f_\epsilon = \left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon \in C_0^\infty$$

where

$$\Phi_\epsilon = \frac{1}{\epsilon}\Phi\left(\frac{x}{\epsilon}\right), f_\epsilon \to f$$

in$H^2_\beta(\mathcal{R})$.ie

$$f_\epsilon \to f\ \ and\ \nabla f_\epsilon \to \nabla f\ in\ L^2_\beta(\mathcal{R}) \qquad (2.14)$$

$$\nabla f_\epsilon = \left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon + \epsilon\left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon \qquad (2.15)$$

It suffices to show

$\left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon \to f\ \ in\ L^2_\beta(\mathcal{R})$ (2.16)

By the lebesgue Dominated convergence Theorem ,we get

$$f.\Phi(\epsilon(.)) \to f\ in\ L^2_\beta(\mathcal{R}) \qquad (2.17)$$

Hence Lemma 1 concludes the proof.
Since $D(\mathcal{R})$ is dense in $H^1_\beta(\mathcal{R})$ $and\ L^2_\beta(\mathcal{R})$, the following lemma follows immediately.

LEMMA 3:

$$H^1_\beta(\mathcal{R}) \subset L^2_\beta(\mathcal{R}) \subset \left(H^1_\beta(\mathcal{R})\right)^{*},$$

from Gelfand triple.

Note. Since $D(\mathcal{R})$ is dense in $H^1_\beta(\mathcal{R})$,the definition of $\langle .,. \rangle$ allows us to interprete the operator $\mathcal{A}$ as a mapping from $H^1_\beta \to \left(H^1_\beta\right)^*$.

for our simplicity,we use

$$V = H^1_\beta(\mathcal{R}), V^* = \left(H^1_\beta(\mathcal{R})\right)^* \text{ and } H = L^2_\beta(\mathcal{R})$$

To use the variational formulation let us defined the following bilinear form on $V \times V$

$$\mathfrak{a}_{(k,\bar{k})}(u,v) = \mathfrak{P}\int_{\mathcal{R}} u'v'e^{-2\beta|x|} + \bar{k}\int_{\mathcal{R}} uve^{-2\beta|x|}dx - \left(\bar{\bar{k}} - \mathfrak{P}\right)\int_{\mathcal{R}} u've^{-2\beta|x|}dx \qquad (2.18)$$

for

$$\mathfrak{P} > 0 \; and \; \bar{k} > 0 \,,$$

One can show $\mathfrak{a}_{(\mathfrak{P},\bar{k})}(u,v)$ is bounded and coercive in V .Define linear operator

$$A_{(\mathfrak{P}\equiv,\bar{k})} : D\left(A_{(\mathfrak{P},\bar{k})}\right) = \{u : u \in V, A_{(\mathfrak{P},\bar{k})}u \in V^*\}$$

intoV^* by

$$\mathfrak{a}_{(\mathfrak{P},\bar{k})}(u,v) = \left(A_{(\mathfrak{P},\bar{k})}, u, v\right)$$

for all $u \in D\left(A_{(\mathfrak{P},\bar{k})}\right)$ for all $v \in V$.

DEFINATION 4.Let X be a Banach space and $a, b \in \beta$ with $a < b, 1 \le p < \infty$.Then $L^2(0,T;X)$ and $L^\infty(0,T\,;X)$denote the space of measurable functions u defined on (a,b) with values in V such that the function $t \to \|u(.,t)\|_X$ is square integrable and essentially bounded. The respective norms are defined by

$$\|u\|_{L^2(0,T;X)} = \left(\int_a^b \|u(.,t)\|_X^2 dt\right)^{\frac{1}{2}} \qquad (2.19)$$

$$\|u\|_{L^\infty(0,T;X)} = ess.sup_{a\le t\le b}\|u(.,t)\|_X. \qquad (2.20)$$

For details on these function space ,see [10]

Definition 5.A function $u: [0,T] \to V$ is a weak solution of (1.5) if

(i) $u \in L^2(0,T;V)$and $u_t \in L^2(0,T;V^*)$;
(ii) For every $v \in V, \langle u_t(t), v\rangle + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u(t),v) = 0$,for t pointwise a.e.in $[0,T]$;

$$u(0) = u_0.$$

Note .The time derivative u_t understood in the distribution sense.The following two lemmas are of critical importance for the existence and uniqueness of the weak solutions.

LEMMA 6.Let $\hookrightarrow H \hookrightarrow V^{*}$ If $u \in L^2(0, T; V)$,$u' \in L^2(0, T; V^{*})$,*thenu* $\in C([0, T]; H)$. Moreover, for any $v \in V$,the real –valued function $t \to \|u(t)\|_H{}^2$ is weakly differentiable in $(0, T)$ and satisfies

$$\frac{1}{2}\frac{d}{dt}\{\|u\|^2\} = \langle u', u\rangle \qquad (2.21)$$

LEMMA 7. (Gronwall’s Lemma) Let $\xi(t)$ be a nonnegative,summable function on $[0, T]$ which satisfies the integral inequality

$$\xi(t) \le C_1 \int_0^t \xi(s)ds + C_2 \qquad (2.22)$$

for constant $C_1 C_2 \ge 0$

almost everywhere $t \in [0. T]$.Then

$\xi(t) \le C_2(1 + C_1 t e^{C_1 t})$a.e on $0 \le t \le T$(2.23).

in particular, if

$\xi(t) \le C_1 \int_0^t \xi(s)ds$ a.e on $0 \le t \le T$,then $\xi(s) = 0$ $a.e$ on $[0, T]$ (2.24)

FOR PROOF SEE [6].

LEMMA 8.The weak solution of (1.5) is unique if it exists.

Proof. Let u_1 and u_2 be two weak solution of (1.5). Let $u = u_1 - u_2$.To prove Lemma 8 .suffices to show that $u = 0$ pointwise a.e.on $[0, T]$.since $\langle u_t(t), v\rangle + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u(t), v) = 0$ for any $v \in V$,we take $v = u \in V$ to get

$$\langle u_t(t), u\rangle + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u(t), u) = 0 \quad (2.25)$$

(2.25) is true point wisea.s .on $[0, T]$.Using (2.1) and the coercivity estimate,we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 \le \gamma\|u\|_H^2, u(0) = 0$$

For some $\gamma > 0$.By Lemma 7, $\|u\|_H = 0$ for all $t \in [0, T]$.Thus $u = 0$ pointwise a.e in $[0, T]$.

To show existence of the weak solution of (1.5) .we first show existence and uniqueness of approximation solution. Now we define the approximate solution of (1.5)

DEFINITION9.A functionu_m: $[0, T] \to V_M$ is an approximate solutions of (1.5) if

(i) $u_m \in L^2(0, T, V_M)$and)$u_{m_t} \in L^2(0, T, V_M)$;

(ii) for every $v \in V_m$ and $[u_{M_t}(t), v]_H + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), v) = 0$ pointwise a.e in $[o, T]$
(iii) $u_M(0) = P_{Mg}$

To prove the existence of approximate solution ,we take $v = u_m$ in

$$[u_t(t), u] + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u(t), u) = 0$$

to get following system of ODEs

$$C^j_{M_t} + \sum_{k=1}^{M} \mathfrak{a}^{jk} C^k_M = 0 \, , C^j_M(0) = g^j \qquad (2.26)$$

Where

$$C^k_M \,\square\, H, C^k_{M_t} \,\square\, H, \text{for} 0 \le t \le T, \mathfrak{a}^{jk}(t) = a(w_j, w_k), \text{and} g^j = (g, w_j)_H \text{for} C: [0, T] \to \mathcal{R}^N,$$

equation 2.24 can be written as

$$\overrightarrow{C}_{M_t} + A(t)\vec{C}_M = 0 \, , \vec{C}_M(0) = \vec{g} \qquad (2.27)$$

Since

$$A \,\square\, L^\infty(0, T; \mathcal{R}^{M\times M}, \text{for} \vec{C}_M = \psi(\vec{C}_M).$$

2.25can be written as

$$\psi(\vec{C}_M(t)) = \vec{g} - \int_0^t A(s)\vec{C}_M(s)\, ds \qquad (2.28)$$

The following lemma is immediate from contraction mapping theorem and (2.28)
LEMMA 10: For any $M \,\square\, N$,there a unique approximate solution $u_m: [0, T] \to V_m$ of (2.28).
The following theorem provide the energy estimate for approximate solutions.

Theorem11.There exist a constant Cdepending only on $T\, and\, \Omega$ such that the approximate solution u_m satisfies

$$\|u_m\|_{L^2(0,T;H)} + \|u_m\|_{L^\infty(0,T;V)} + \|u_{m_t}\|_{L^2(0,T;H)} \le C\|g\|_H \qquad (2.29)$$

Proof: For every $v \,\square\, u_m$ we have

$[u_{M_t}(t), v]_H + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), v) = 0.$
take$v \,\square\, u_m(t)$,then we have

$$[u_{M_t}(t), v]_H + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), v) = 0., \text{pointwisea.e in } (0, T) \qquad (2.30)$$

using 2.30 and the coercivity estimate.We find that there exists constants

$$\rho > 0, \gamma > 0$$

such that

$$\frac{1}{2}\frac{d}{dt}(e^{-2\gamma t}\|u_M\|_H^2) + \rho e^{-2\gamma t \|u_M\|_V^2} \le 0 \qquad (2.31)$$

Integrating 2.31 with respect to t,using the initial condition $u_M(0) = P_m(g)$, (2.39).

and

$\|P_m(g)\|_H \leq \|g\|_H$, we get

$$\frac{1}{2}\frac{d}{dt}(e^{-2\gamma t}\|u_M\|_H^2) + \rho e^{-2\gamma t\|u_M\|_V^2} \quad (2.32)$$

taking the supriemum over $[0,T]$,we get

$$\|u_m\|_{L^2(0,T;H)} + \|u_m\|_{L^2(0,T;V)} \leq C\|g\|_H^2 \quad (2.33)$$

Since

$u_{M_t}(t) \in V_M^*$,

we have

$$\|u_{M_t}(t)\|_{V^*} = \sup_{v \in V_M^*} \frac{(u_{M_t}(t),v)H}{\|v\|_V}, v \neq 0 \quad (2.34)$$

Using the notion of approximate solution and boundedness of A we have

$$\|u_m\|_{L^\infty(0,T;H)} + \|u_m\|_{L^2(0,T;V)} + \|u_{m_t}\|_{L^2(0,T;H)} \leq C\|g\|_H \quad (2.35)$$

To complete the proof of weak solution, we now show the convergence of the approximate solutions by using weak compactness argument.

DEFINITION 12: Let $L^2(0,T;V^*)$ *be* the dual space of $L^2(0,T;V)$.Let $f \in L^2(0,T;V^*)$ and $u \in L^2(0,T;V)$,then we say $u_M \to u \; in \; L^2(0,T;V)$ weakly if

$$\int_0^T \langle f(t), u_M(t)\rangle dt \to \int_0^T \langle f(t), u(t)\rangle dt \quad \forall \; f \in L^2(0,T;V^*) \quad (2.36)$$

Lemma 13 .A subsequence $\{u_m\}$ of approximate solutions u_m converge weakly in $L^2(0,T;V^*)$ to a weak solution $u \in C([0,T];H) \cap L^2(0,T;V)$of (1.5) with $u_t \in L^2(0,T;V^*)$.Moreover,it satisfies

$$\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;H)} \leq C\|g\|_H \quad (2.37)$$

PROOF.Theorem 11 implies that the approximate solutions $\{u_m\}$ are bounded in $L^2(0,T;V)$ and their derivatives $\{u_{m_t}\}$ are bounded in $L^2(0,T;V^*)$. By the Banach-Alaoglu theorem, we can extract a subsequence $\{u_m\}$ such that

$$u_m \to u \; in \; L^2(0,T;V), \; u_{m_t} \to u_{t \; in} \; L^2(0,T;V^*) \text{weakly} (2.38)$$

Let $\phi \in C_0^\infty$(0,T) be a real-valued test function and $w \in V_N$ for some $N = \mathcal{N}$.Replacing v by $\phi(t)w \; in \; \langle u_{M_t}(t), v\rangle_H + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), v) = 0$

and integrating from 0 toT, we get.

$$\int_0^T \langle u_{M_t}(t), \phi(t)w\rangle_H dt + \int_0^T \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), \phi(t)w)dt = 0 \; for \; M \geq N \; 2.38$$

taking the limit as $M \to \infty$,we get

$$\int_0^T (u_{M_t}(t), \phi w)_H dt = \int_0^T \langle u_t, \phi w\rangle dt \quad (2.39)$$

by using boundedness of $\mathfrak{a}_{(\mathfrak{P},\bar{k})}$,we get

$$\int_0^T \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u_M(t), \phi(t)w)dt = \int_0^T \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u(t), \phi(t)w)\, dt \quad (2.40)$$

using boundedness of $\mathfrak{a}_{(\mathfrak{P},\bar{k})}$,we get

$$\langle u_t(t), w\rangle + \mathfrak{a}_{(\mathfrak{P},\bar{k})}(u, w) = 0 \quad (2.41)$$

pointwise a.e in $(0, T)$

since 2.41 is true for all $w \in V_M$

$U_{M \in N} V_M$and(2.42)

is dense in V,so (2.42) holds for all $w \in V$. now it remains to show that $u(0) = u_0$.using (2.42),integrating by parts and Galerkin approximation we have

$$\langle u(0), w\rangle = \langle u_0, w\rangle \; as \;\; M \to \infty$$

for every $w \in V_M$.Thus $u(0) = u_0$

3 EXISTENCEOF OPTIMALPARAMETER

Lemma 14 .Let $v \in V$ .Then the mapping $(\mathfrak{P}, \bar{k}) \to A_{\mathfrak{P},\bar{k}} v$ from

$$\mathcal{P}_{ad} = \{q = (\mathfrak{P}, \bar{k}) \in [\mathfrak{P}_{min}, \mathfrak{P}_{max}] \times [\bar{k}_{min}, \bar{k}_{max}]\}$$

intoV' is continuous.

Proof .Suppose that$q_n \to q$ in $\mathcal{R}^2$ as $n \to \infty$.We denote $A = A_{\mathfrak{P},\bar{k}}$ and $A_n = A_{\mathfrak{P}_n,\bar{k}_n}$.We claim that

$$\|(A_n - A)v\|_{V'} \to 0$$

as $n \to \infty$. Let $w \in V$ with$\|w\| \leq 1$.Then

$$\begin{aligned} |\langle(A_n - A)v, w\rangle|^2 &\leq \int_{\mathcal{R}} (|\mathfrak{P}_n - \mathfrak{P}||u'||w'|dx)^2 + \left(\int_{\mathcal{R}} |\bar{k}_n - k||u||w|dx\right)^2 + \left(\int_{\mathcal{R}} |\bar{k}_n - k||u'||w|dx\right) \\ &+ \int_{\mathcal{R}} (|\mathfrak{P}_n - \mathfrak{P}||u'||w|dx)^2 \\ &\leq 2|\mathfrak{P}_n - \mathfrak{P}|^2 \int_{\mathcal{R}} |u'|(x)^2 dx + |\bar{k}_n - k|^2 \int_{\mathcal{R}} |u'|(x)^2 dx + |\bar{k}_n - k|^2 \int_{\mathcal{R}} |u'|(x)^2 dx \to 0 \\ & as\; n \to \infty \end{aligned}$$

Lemma 15.Suppose that $\mathfrak{P}_n, \bar{k}_n \to \mathfrak{P}, \bar{k}$in $\mathcal{R}^2$, and $v_n \to v$ weakly in V as $n \to \infty$.Then $A_n v_n \to A_v$ weakly in v'.

Proof.Let $w \in V$,then.

$$\left|\langle A_{n,} v_{n,} w\rangle - \langle A_v, w\rangle\right| = \left|\langle A_n w, v_{n,}\rangle - \langle A_w v\rangle\right| \le |\langle A_n - A\rangle w, v_n| + |\langle Aw, v_n - v\rangle| \quad (2.43)$$

Since a weakly convergent sequence is bounded, we have

$$|\langle A_n - A\rangle w, v_n| \le \|A_n w - Aw\|V' \|v_n\| \le c\|A_n w - Aw\|V' \to 0$$

as$n \to \infty$ Lemma 14.The second term

$$\left|\langle A_{n,} v_{n,} - v\rangle\right| \to 0$$

since$v_n \to v$ weakly.

Lemma16.Let $q_n \in \mathcal{P}_{ad}$. Then the solution map $q \to u(q)$ from $\mathcal{P}$ into $C([0,T];H)$ is continuous.

Proof.Let$q_n \to q$ in q_{ad} as $n \to \infty$.Since$U(t;q)$ is the weak solution of (1.5) for any$q \in \mathcal{P}_{ad}$ we have the following estimate.

$$\|u_M(t;q_n)\|_{L^\infty(0,T;H)} + \|u_M(t;q_n),\|_{L^2(0,T;V)} + \left\|u_{M_t}(t;q_n)\right\|_{L^2(0,T;H)} \le C\|g\|_H \quad (2.44)$$

Where C is constant independent of $q \in \mathcal{P}_{ad}$. Estimate (2.44) shows that $U(t;q)$ is bounded in $W(0,T)$.Since$W(0,T)$ is reflexive.we can choose a sub-sequence $u(t;q_{n_k})$ weakly convergent to a function z in $W(0,T)$.The fact that $u_M(t;q_n)$ is bounded in $W(0,T)$ implies that $u_M(t,q_n)$ is bounded in $L^2(0,T;V)$,so $u(t;q_{n_k})$weakly convergent to a function z in $L^2(0,T;H)$.Since V is compactly imbedded in H,then by the classical compactness theorem[4] $u(t;q_n) \to z$ in $L^{\infty 2}(0,T;H)$,.By (2,44) the derivative $u'(t;q_{n_k})$ and z' are uniformly bounded in $L^\infty(0,T;H)$.Therefore functions $\{u(t;q_{n_k}), z\}_{k=1}^{\infty}$ are equicontinuous in $C([0,T];H)$..Thus $u(t;q_{n_k}) \to z$ in $C([0,T];H)$ ...In particular $u(t;q_{n_k}) \to z(t)$ in H and $u(t;q_{n_k}) \to z$ weakly in V for any $t \in [0,T]$.By lemma 15,$A_{n_k}]\, u(t;q_n) \to Az(t)$ weakly in V'.Now we see that z satisfies the equation given in definition 5,ie it is the weak solution $u(q)$ The uniqueness of the weak solution implies that $u(q_n) \to u(q) as\ n \to \infty$ in $C([0,T];H)$ for the entire sequence $u(q_n)$ and not for its subsequence. Thus that $u(t;q_n) \to u(q)$ in $C([0,T];H)$ as that $q_n \to q$ in P as claimed.

3. CONCLUSIONS

The Black-Scholes option pricing model with transaction cost was discussed, where we use an adjusted volatility given as $\widehat{\sigma}^2 = \sigma^2(1 - Lesgn(\partial_S^2 v)$ and a continuous random work which generalizes the works in the literature.The parameters associated with the Black –Scholes option pricing model with transaction cost was considered. Also the existence and uniqueness of weak solution of Black-Scholes option price with transaction cost was studied. The continuity of weak solution of the parameters was discussed and similar solution as in literature obtained. The extra terms introduced in this paper is to directly model asset pricedynamics in the case when the large trader chooses a givenstock-trading strategy.If transaction costs are taken into account perfect replicationof the contingent claim is no longer possible. Hence, one can re-adjust the volatility (when the investor’s preferences are characterized by anexponential utility function)in the form;

$$\sigma^2 = \hat{\sigma}^2 \left(1 + \varphi\left(a^2 e^{-r(t-T)} S^2 \partial_S^2 V\right)\right)^2,$$

and if a the weak solution can be obtained.

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