A POSITIVE INTEGER N SUCH THAT $p_n + p_{n+3} \sim p_{n+1} + p_{n+2}$ FOR ALL $n \geq N$

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ABSTRACT

According to Bertrand's postulate,we have $p_n + p_n \geq p_{n+1}$. Is it true that for all $n > 1$ then $p_{n-1} + p_n \geq p_{n+1}$? Then $p_n + p_{n+3} > p_{n+1} + p_{n+2}$where $n \geq N$, N is a large enough value?

KEYWORDS

Bertrand's Postulate, Rosser's theorem, L'Hospital Rule, prime number.

1. INTRODUCTION

In 1845, Bertrand conjectured what became known as Bertrand's postulate: twice any prime strictly exceeds the next prime. Tchebichef presented his proof of Bertrand's postulate in 1850 and published it in 1852. It is now sometimes called the Bertrand-Chebyshev theorem. Surprisingly, a stronger statement seems not to be well known, but is elementary to prove: The sum of any two consecutive primes strictly exceeds the next prime, except for the only equality 2 + 3 = 5. After I conjectured and proved this statement independently, a very helpful referee pointed out that Ishikawa published this result in 1934 (with a different proof). This observation is a special case of a much more general result, Theorem 1, that is also elementary to prove (given the prime number theorem), and perhaps not previously noticed: Ifp_n denotes the nth prime, $n = 1, 2, 3, \ldots$ with $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$, and then there exists a positive integer N such that $p_n + p_{n+3} \sim p_{n+1} + p_{n+2}$for ll $n \geq N$. We give the following result.

2. MAIN RESULT

Theorem 1. Ifn are nonnegative integers, then there exists a large enough positive integer N such that, for all $n \geq N, p_n + p_{n+3} \sim p_{n+1} + p_{n+2}$.

Applying Rosser's theorem for all $n \geq 6$, we have

$$n(\ln n + \ln\ln n - 1) < p_n < n(\ln n + \ln\ln n)$$

$$(n+1)[\ln(n+1) + \ln\ln(n+1) - 1] < p_{n+1} < (n+1)[\ln(n+1) + \ln\ln(n+1)]$$
$$(n+2)[\ln(n+2) + \ln\ln(n+2) - 1] < p_{n+2} < (n+2)[\ln(n+2) + \ln\ln(n+2)]$$
$$(n+3)[\ln(n+3) + \ln\ln(n+3) - 1] < p_{n+3} < (n+3)[\ln(n+3) + \ln\ln(n+3)]$$

Consider the expression

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$$A = \frac{n(\ln n + \ln\ln n - 1) + (n+3)[\ln(n+3) + \ln\ln(n+3) - 1]}{(n+1)[\ln(n+1) + \ln\ln(n+1)] + (n+2)[\ln(n+2) + \ln\ln(n+2)]}$$

We consider the following limit

$$B = \lim_{n\to+\infty} \frac{\ln n + \ln\ln n + \frac{1}{\ln n} + \ln(n+3) + \ln\ln(n+3) + \frac{1}{\ln(n+3)}}{\ln(n+1) + \ln\ln(n+1) + 2 + \frac{1}{\ln(n+1)} + \ln(n+2) + \ln\ln(n+2) + \frac{1}{\ln(n+2)}}$$

Taking the ln of the numerator and denominator and applying L'Hospital Rule gives

$$\lim_{n\to+\infty} n(\ln n + \ln\ln n - 1)$$

$$= \lim_{n\to+\infty} \ln n + \ln\ln n - 1 + n\left(\frac{1}{n} + \frac{\frac{1}{n}}{\ln n}\right)$$

$$= \lim_{n\to+\infty} \ln n + \ln\ln n + \frac{1}{\ln n}$$

$$\lim_{n\to+\infty} n\,[\ln(n+3) + \ln\ln(n+3) - 1]$$

$$= \lim_{n\to+\infty} \ln(n+3) + \ln\ln(n+3) - 1 + (n+3)\left(\frac{1}{n+3} + \frac{\frac{1}{n+3}}{\ln(n+3)}\right)$$

$$= \lim_{n\to+\infty} \ln(n+3) + \ln\ln(n+3) + \frac{1}{\ln(n+3)}$$

$$\lim_{n\to+\infty} n\,[\ln(n+1) + \ln\ln(n+1)]$$

$$= \lim_{n\to+\infty} \ln(n+1) + \ln\ln(n+1) + (n+1)\left(\frac{1}{n+1} + \frac{\frac{1}{n+1}}{\ln(n+1)}\right)$$

$$= \lim_{n\to+\infty} \ln(n+1) + \ln\ln(n+1) + \frac{1}{\ln(n+1)} + 1$$

$$\lim_{n\to+\infty} n\,[\ln(n+2) + \ln\ln(n+2)]$$

$$= \lim_{n\to+\infty} \ln(n+2) + \ln\ln(n+2) + (n+2)\left(\frac{1}{n+2} + \frac{\frac{1}{n+2}}{\ln(n+2)}\right)$$

$$= \lim_{n\to+\infty} \ln(n+2) + \ln\ln(n+2) + \frac{1}{\ln(n+2)} + 1$$

Then we see

$$B = \lim_{n\to+\infty} \frac{\ln n + \ln\ln n + \frac{1}{\ln n} + \ln(n+3) + \ln\ln(n+3) + \frac{1}{\ln(n+3)}}{\ln(n+1) + \ln\ln(n+1) + 2 + \frac{1}{\ln(n+1)} + \ln(n+2) + \ln\ln(n+2) + \frac{1}{\ln(n+2)}}$$

When $n \to +\infty$ then

$$B = \lim_{n\to+\infty} \frac{\ln n + \ln(n+3)}{\ln(n+1) + \ln(n+2)} = \lim_{n\to+\infty} \frac{\ln(n^2+3n)}{\ln(n^2+3n+2)} = 1$$

(Because $n^2 + 3n \sim n^2 + 3n + 2,$ for $n \to +\infty$)

Or, for $n \geq N, N$ is a large enough positive integer, then$A \sim 1,$

$$\frac{n(\ln n + \ln\ln n - 1) + (n+3)[\ln(n+3) + \ln\ln(n+3) - 1]}{(n+1)[\ln(n+1) + \ln\ln(n+1)] + (n+2)[\ln(n+2) + \ln\ln(n+2)]} \sim 1$$

It turns out,$\frac{p_n+p_{n+3}}{p_{n+1}+p_{n+2}} \sim 1$, or have mean is $p_n + p_{n+3} \sim p_{n+1} + p_{n+2}$.

3. CONCLUSION

In short, that is also elementary to prove (given the prime number theorem), and perhaps not previously noticed: If p_n denotes the nth prime, $n = 1,2,3,...$ with $p_1 = 2$, $p_2 = 3$, $p_3 = 5,...$, and then there exists a positive integer N such that $p_n + p_{n+3} \sim p_{n+1} + p_{n+2}$ *for 11n* $\geq N$. But can not confirm that $p_n + p_{n+3} = p_{n+1} + p_{n+2}$ where $n \geq N$, N is a large enough value?

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