EXPONENTIATED TRANSMUTED RAYLEIGH DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this present article, we have proposed a new probability distribution known as Exponentiated transmuted Rayleigh distribution and studied some statistical properties of the proposed model. Further, for application point of view, we have derived it from Rayleigh distribution as a baseline distribution and proved its application in comparison to the its sub-models in terms of fitting a real data as well as simulated data through Akaike's Information Criteria (AIC) and Bayesian Information Criteria (BIC) and log likelihood (LL) criterion of goodness of fit.

KEYWORDS

Rayleigh distribution, Transmuted Rayleigh Distribution, Exponentiated distribution, Simulation Study and real life data.

1. INTRODUCTION

In recent times, various families of distributions have been proposed through expanding general families by different techniques. The statistical literature includes number of new families of distributions proposed by various researchers, for instance, Gupta and Kundu [6] first developed a generalization of the standard exponential distribution, called the exponentiated exponential distribution (EED), Elbatal and Muhammed [4] introduced the exponentiated generalized inverse Weibull distribution and one more example is the quadratic rank transmutation map (QRTM) technique defined by Shaw and Buckley [8]. Ahmad et al. [1] defined the transmuted inverse Rayleigh distribution and studied its various properties while Dey et al. [3] considered the different estimation methods of transmuted Rayleigh distribution and derived its Statistical Properties. Fatima and Ahmad [5] discussed the exponentiated Invert exponential distribution and obtained its different structural properties. In addition, for instance see Kareema and Ashraf [7] studied the exponentiated transmuted exponential distribution and discussed its important properties. More recently, Uzma et al. [9] proposed the transmuted generalized Inverse Rayleigh distribution and derived its some characteristic properties.

The density function of Rayleigh distribution (RD) is given by:

$$g(x) = 2\theta x e^{-\theta x^2} \quad ; x > 0, \theta > 0. \tag{1.1}$$

The corresponding cdf of RD is given by

$$G(x) = 1 - e^{-\theta x^2} \quad ; x > 0 \,, \theta > 0 \,, \tag{1.2}$$

where θ is the scale parameter.

2. Exponentiated Transmuted Rayleigh Distribution

If X follows the transmuted distribution then its cdf is given as:

$$F^{\#}(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 \quad , \quad |\lambda| \le 1 \tag{2.1}$$

where $G(x)$ is the cdf of the base distribution. It must be noted that as $\lambda = 0$, the proposed distribution reduces to base distribution.

If X follows an Exponentiated distribution then its cdf is given as:

$$F(x) = \left(F^{\#}(x)\right)^{\beta} \tag{2.2}$$

The cdf of Exponentiated transmuted Rayleigh distribution (ETRD) with parameters θ, λ and β is given by:

$$F\,(x) = \left((1-e^{-\theta x^2})(1+\lambda e^{-\theta x^2})\right)^{\beta} \quad ; x>0, \theta, \lambda, \beta > 0. \tag{2.3}$$

Different possible shapes for the distribution function of ETRD are given in figure 1 respectively.

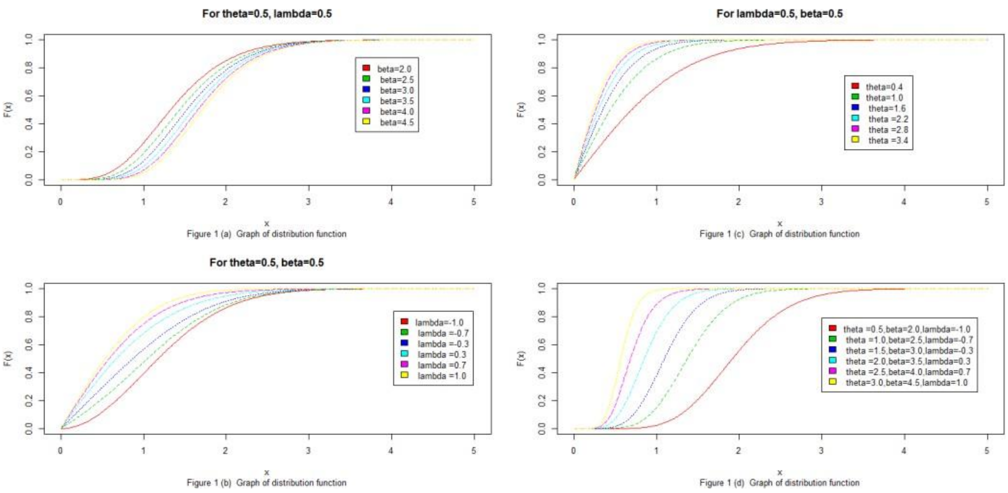

Figure 1: Graphs of the distribution Function.

Differentiating equation (2.2) with respect to x gives the pdf of the Exponentiated model as

$$f\,(x) = 2\theta\beta x e^{-\theta x^2}\left((1-\lambda) + 2\lambda e^{-\theta x^2}\right)\left((1-e^{-\theta x^2})(1+\lambda e^{-\theta x^2})\right)^{\beta-1} ; x>0, \theta, \lambda, \beta > 0. \tag{2.4}$$

where θ, β and λ are scale, shape and transmuted parameters respectively.

Figure 2 gives different plots of the ETRD curves for various values of the parameters θ, λ and β. Plots of hazard rate function of ETRD for selected parameter values are shown in Figure 3.

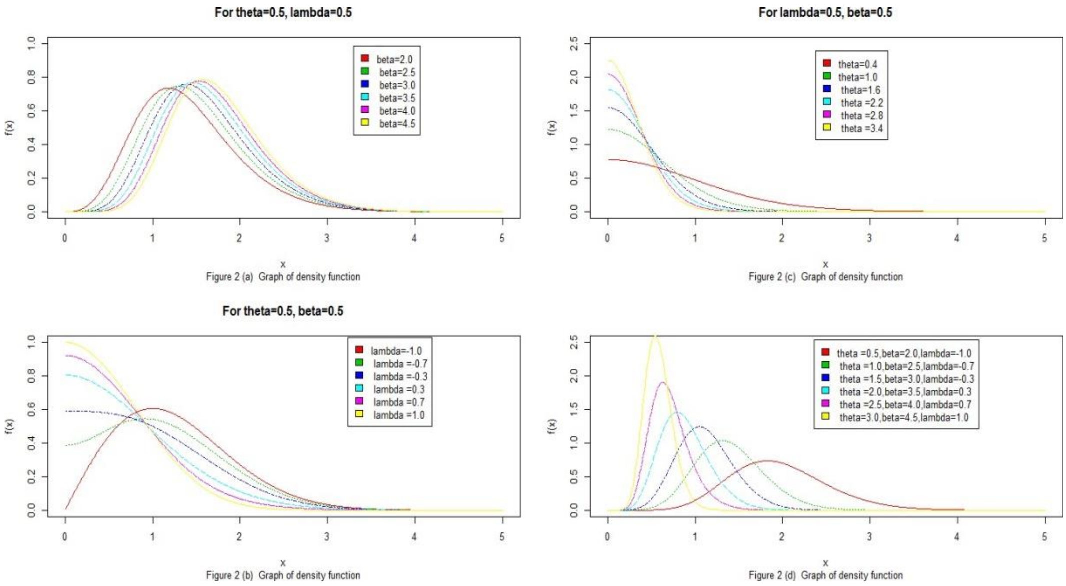

Figure 2: Graphs of the ETRD.

3. RELATIONSHIP WITH OTHER DISTRIBUTIONS

Different important theoretical models can be obtained from the proposed ETRD when its parameters are changed. The ETRD has 6 special cases which are given below:
Different important theoretical models can be obtained from the proposed ETRD as:

- For $\beta = 1$, Equation (2.4) reduces to give the Transmuted Rayleigh distribution (TRD) with pdf as:

$$f(x) = 2\theta x e^{-\theta x^2}\left((1-\lambda) + 2\lambda e^{-\theta x^2}\right) ; x > 0, \theta, \lambda > 0. \tag{3.1}$$

- For $\beta = 1$and $\lambda = 0$, Equation (2.4) reduces to give the one parameter Rayleigh distribution (RD) with pdf as:

$$f(x) = 2\theta\beta x e^{-\theta x^2} ; x > 0, \theta > 0. \tag{3.2}$$

- For $\lambda = 0$, Equation (2.4) reduces to give the Exponentiated Rayleigh distribution (ERD) with pdf as:

$$f(x) = 2\theta\beta x e^{-\theta x^2}\left(1 - e^{-\theta x^2}\right)^{\beta-1} ; x > 0, \theta, \beta > 0. \tag{3.3}$$

- For $\theta = 1$, Equation (2.4) reduces to give the Exponentiated transmuted Standard Rayleigh distribution (ETSRD) with pdf as:

$$f(x) = 2\beta x e^{-x^2}\left((1-\lambda) + 2\lambda e^{-x^2}\right)\left((1 - e^{-x^2})(1 + \lambda e^{-x^2})\right)^{\beta-1} ; x > 0, \lambda, \beta > 0. \tag{3.4}$$

- For $\lambda = 0, \theta = 1$, Equation (2.4) reduces to give the Exponentiated Standard Rayleigh distribution (ESRD) with pdf as:

$$f(x) = 2\beta x e^{-x^2}\left(1 - e^{-x^2}\right)^{\beta-1}; x > 0, \beta > 0. \tag{3.5}$$

- For $\beta = 1, \theta = 1$ Equation (2.4) reduces to give the Transmuted Standard Rayleigh distribution (TSRD) with pdf as:

$$f(x) = 2xe^{-x^2}\left((1-\lambda) + 2\lambda e^{-x^2}\right); x > 0, \lambda > 0. \tag{3.6}$$

- For $\lambda = 0, \theta = 1$and $\beta = 1$, Equation (2.4) reduces to give the Standard Rayleigh distribution (SRD) with pdf as:

$$f(x) = 2xe^{-x^2} \quad ; x > 0. \tag{3.7}$$

4. Reliability Analysis

The reliability function of ETRD is defined by:

$$R(x) = 1 - F(x) = 1 - \left(1 - e^{-\theta x^2}\right)\left(1 + \lambda e^{-\theta x^2}\right)^{\beta} \quad ; x > 0, \theta, \lambda, \beta > 0. \tag{4.1}$$

The hazard function of ETRD is defined by:

$$h(x) = \frac{f(x)}{R(x)} = \frac{2\theta\beta x e^{-\theta x^2}\left((1-\lambda) + 2\lambda e^{-\theta x^2}\right)\left(1 - e^{-\theta x^2}\right)\left(1 + \lambda e^{-\theta x^2}\right)^{\beta-1}}{1 - \left(1 - e^{-\theta x^2}\right)\left(1 + \lambda e_{-\theta x^2}\right)^{\beta}}; \; x, \theta, \lambda, \beta > 0. \tag{4.2}$$

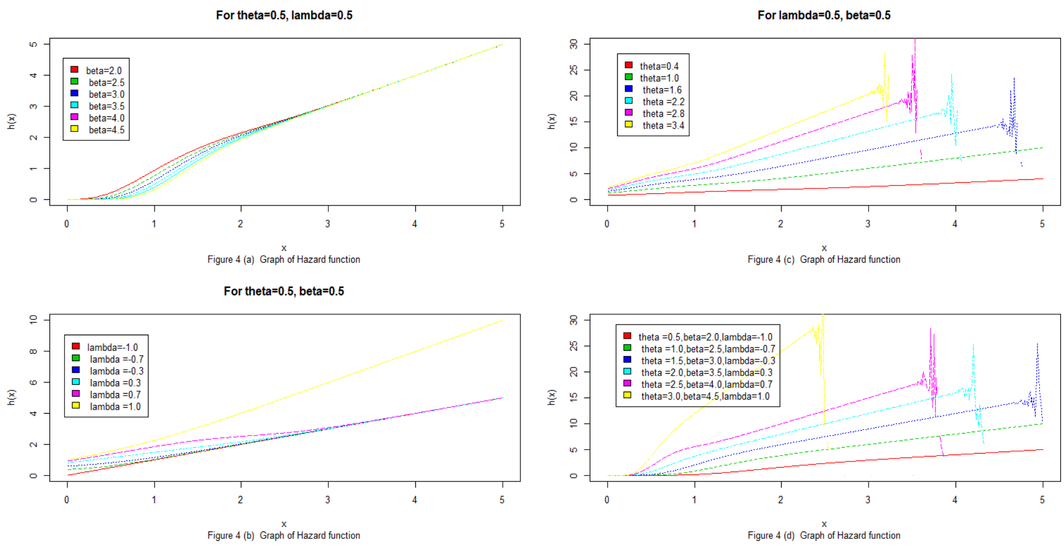

Figure 3: Graphs of the Hazard Function.

The reverse hazard rate of ETRD is given as:

$$\phi(x) = \frac{f(x)}{F(x)} = \frac{2\theta\beta x e^{-\theta x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)}{\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)} \quad ; x>0, \theta, \lambda, \beta > 0. \tag{4.3}$$

The cumulative hazard function of the ETR model is denoted by $H(x)$ and is given as:

$$H(x) = -\ln\left(1-F(x)\right) = -\ln\left(1-\left(\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)\right)^{\beta}\right) \quad ; x>0, \theta, \lambda, \beta > 0. \tag{4.4}$$

5. Statistical Properties of the ETRD

Moments of the ETRD

In this sub section we study the moment about the mean and the moment about the origin of the ETRD.

Theorem 5.1: The r[th] moment about the mean of ETRD is given as follows:

$$E(X-\mu)^r = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{r}{k}\binom{1}{m} \times \frac{\Gamma\left((r-k)/2\right)+1}{\theta^{(r-k)/2}(i+j+m+1)^{((r-k)/2)+1}}. \tag{5.1}$$

and the moment about the origin is

$$E(X)^r = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1} (-1)^i 2^m \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m} \times \frac{\Gamma\left(1+r/2\right)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}. \tag{5.2}$$

Proof: Using equation (2.4), the r[th] moment is given by

$$E(X-\mu)^r = \int_0^{\infty} (X-\mu)^r f(x)dx$$

$$= \int_0^{\infty} (X-\mu)^r 2\theta\beta x e^{-\theta x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{\beta-1} dx \tag{5.3}$$

Using the series expansion of $\left(1-e^{-\theta x^2}\right)^{\beta-1}$, $\left(1+\lambda e^{-\theta x^2}\right)^{\beta-1}$, $(X-\mu)^r$ and $\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)$

$$\left(1-e^{-\theta x^2}\right)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} e^{-i\theta x^2}$$

$$\left(1+\lambda e^{-\theta x^2}\right)^{\beta-1} = \sum_{j=0}^{\infty} \binom{\beta-1}{j} \lambda^j e^{-j\theta x^2}$$

$$(X-\mu)^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} x^{r-k} \mu^k$$

$$\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)^{1} = \sum_{m=0}^{1} \binom{1}{m} (1-\lambda)^{(1-m)} (2\lambda)^m e^{-m\theta x^2}$$

Expression (5.3) takes the following form:

$$E(X-\mu)^r = 2\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{r}{k}\binom{1}{m}$$

$$\times \int_0^{\infty} x^{r-k+1} e^{-(i+j+m+1)\theta x^2} dx.$$

$$E(X-\mu)^r = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{r}{k}\binom{1}{m}$$

$$\times \frac{\Gamma\left(((r-k)/2)+1\right)}{\theta^{(r-k)/2}(i+j+m+1)^{((r-k)/2)+1}}$$

Now, if we put $\mu = 0$ .So, the moment about the origin is

$$E(X)^r = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1} (-1)^i 2^m \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m}$$

$$\times \frac{\Gamma(1+r/2)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}.$$

Hence Proved.

Remarks:

- If we put r=1, we get the mean of ETRD as

$$E(X-\mu) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{1}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{k}\binom{1}{m}$$

$$\times \frac{\Gamma\left((3-k)/2\right)}{\theta^{(1-k)/2}(i+j+m+1)^{(3-k)/2}}.$$

- If we put r=2, we get the variance of ETRD as

$$E(X-\mu)^2 = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{2}{k}\binom{1}{m}$$

$$\times \frac{\Gamma\left(((2-k)/2)+1\right)}{\theta^{(2-k)/2}(i+j+m+1)^{((2-k)/2)+1}}$$

- If we put r=3, we get the third moment of ETRD as

$$E(X-\mu)^3=\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{3}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{3}{k}\binom{1}{m}\times\frac{\Gamma((3-k)/2)+1}{\theta^{(3-k)/2}(i+j+m+1)^{((3-k)/2)+1}}.$$

- If we put r=4, we obtain the fourth moment of ETRD as

$$E(X-\mu)^4=\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{4}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{4}{k}\binom{1}{m}\times\frac{\Gamma((4-k)/2)+1}{\theta^{(4-k)/2}(i+j+m+1)^{((4-k)/2)+1}}.$$

- The Coefficient of Variation is given by

$$CV=\frac{\sqrt{Var(x)}}{E(X-\mu)}$$

$$=\frac{\sqrt{\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{2}{k}\binom{1}{m}\times\frac{\Gamma((2-k)/2)+1}{\theta^{(2-k)/2}(i+j+m+1)^{((2-k)/2)+1}}}}{\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{1}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{k}\binom{1}{m}\times\frac{\Gamma((3-k)/2)}{\theta^{(1-k)/2}(i+j+m+1)^{(3-k)/2}}}.$$

- The Coefficient of skewness is given by

$$CS=\frac{E(X-\mu)^3}{\left(E(X-\mu)^2\right)^{\frac{3}{2}}}$$

$$=\frac{\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{3}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{3}{k}\binom{1}{m}\times\frac{\Gamma((3-k)/2)+1}{\theta^{(3-k)/2}(i+j+m+1)^{((3-k)/2)+1}}}{\left(\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{(i+k)}2^m\mu^k\lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{2}{k}\binom{1}{m}\times\frac{\Gamma((2-k)/2)+1}{\theta^{(2-k)/2}(i+j+m+1)^{((2-k)/2)+1}}\right)^{\frac{3}{2}}}.$$

- The Coefficient of kurtosis is given by

$$CK = \frac{E(X-\mu)^4}{\left(E(X-\mu)^2\right)^2}$$

$$CK = \frac{\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{4}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{4}{k}\binom{1}{m} \times \frac{\Gamma\left((4-k)/2\right)+1}{\theta^{(4-k)/2}(i+j+m+1)^{((4-k)/2)+1}}}{\left(\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1} (-1)^{(i+k)} 2^m \mu^k \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{2}{k}\binom{1}{m} \times \frac{\Gamma\left((2-k)/2\right)+1}{\theta^{(2-k)/2}(i+j+m+1)^{((2-k)/2)+1}}\right)^2}.$$

Moment Generating Function

Theorem 5.2: Let X have an ETRD. Then MGF of X denoted by $M_X(t)$ is given by:

$$M_X(t) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}\sum_{r=0}^{\infty} \frac{t^r}{r!} (-1)^i 2^m \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m} \times \frac{\Gamma(1+r/2)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}. \tag{5.4}$$

Proof: -By definition

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)dx.$$

Using Taylor series expansion, we get

$$M_X(t) = \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\cdots\right) f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)dx. = \sum_{r=0}^{\infty}\frac{t^r}{r!} E\left(X^r\right)$$

$$M_X(t) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}\sum_{r=0}^{\infty} \frac{t^r}{r!} (-1)^i 2^m \lambda^{(m+j)} (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m} \times \frac{\Gamma(1+r/2)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}.$$

Hence Proved.

Characteristic Function

Theorem 5.3: Let X have an ETRD. Then characteristic function of X denoted by $\phi_X(t)$ is given by:

$$\phi_X(t) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}\sum_{r=0}^{\infty} \frac{(it)^r}{r\,!} (-1)^{i}\, 2^{m}\, \lambda^{(m+j)} \quad (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m}$$
$$\times \frac{\square\Gamma(1+r/2)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}. \qquad (5.5)$$

Proof: -By definition

$$\phi_X(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)dx.$$

Using Taylor series expansion, we get

$$\phi_X(t) = \int_0^{\infty}\left(1 + itx + \frac{(itx)^2}{2!} + \cdots\right) f(x)dx.$$
$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)$$
$$\phi_X(t) = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}\sum_{r=0}^{\infty} \frac{(it)^r}{r\,!} (-1)^{i}\, 2^{m}\, \lambda^{(m+j)} \quad (1-\lambda)^{(1-m)} \binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m}$$
$$\times \frac{\square\Gamma(1+r/2)}{\theta^{r/2}(i+j+m+1)^{1+r/2}}.$$

Hence proved.

Random Sample Generator

By using the method of inversion, we can generate random numbersfrom ETRD.

Let $F(x) = u,\quad$ where $u \sim U(0,1)$

$$\therefore \qquad u = \left(1 - e^{-\theta x^2}\right)\left(1 + \lambda e^{-\theta x^2}\right)^{\beta}. \qquad (5.6)$$

On solving equation (5.6) for x in terms of u, we get:

$$x = \sqrt{\left(-\frac{1}{\theta}\ln\left(1 - \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u^{1/\beta}}}{2\lambda}\right)\right)}. \qquad (5.7)$$

Harmonic mean of ETR distribution

The harmonic mean (H) is given as:

$$\frac{1}{H}=E\left(\frac{1}{X}\right)=\int_0^{\infty}\frac{1}{x}\,2\theta\beta x e^{-\theta x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{\beta-1}dx$$

The above equation takes the following form:

$$\frac{1}{H}=2\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i 2^m \lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m}\times\int_0^{\infty}e^{-(i+j+m+1)\theta x^2}dx.$$

$$\frac{1}{H}=\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i 2^m \lambda^{(m+j)}(1-\lambda)^{(1-m)}\binom{\beta-1}{i}\binom{\beta-1}{j}\binom{1}{m}\left(\frac{\theta\pi}{(i+j+m+1)}\right)^{1/2}. \tag{5.8}$$

6. RENYI ENTROPY

Renyi entropy is defined by

$$I_R(\rho)=\frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty}f(x)^{\rho}dx\right\}, \tag{6.1}$$

where $\rho>0$ and $\rho\neq 1$.

If X has ETRD $(x;\beta,\lambda,\theta)$, then by putting equation (2.4) in (6.1) we have:

$$I_R(\rho)=\frac{1}{1-\rho}\log\left[\int_0^{\infty}2^{\rho}\theta^{\rho}\beta^{\rho}x^{\rho}e^{-\theta\rho x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)^{\rho}\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{\rho(\beta-1)}dx\right] \tag{6.2}$$

Let $u(x)=\int_0^{\infty}f(x)^{\rho}dx$

$$\therefore u(x)=\int_0^{\infty}2^{\rho}\theta^{\rho}\beta^{\rho}x^{\rho}e^{-\theta\rho x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)^{\rho}\left(1-e^{-\theta x^2}\right)^{\rho(\beta-1)}\left(1+\lambda e^{-\theta x^2}\right)^{\rho(\beta-1)}dx.$$

Using the series expansion of $\left(1-e^{-\theta x^2}\right)^{\rho(\beta-1)}$, $\left(1+\lambda e^{-\theta x^2}\right)^{\rho(\beta-1)}$ and $\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)^{\rho}$

$$\left(1-e^{-\theta x^2}\right)^{\rho(\beta-1)}=\sum_{i=0}^{\infty}(-1)^i\binom{\rho(\beta-1)}{i}e^{-i\theta x^2}$$

$$\left(1+\lambda e^{-\theta x^2}\right)^{\rho(\beta-1)}=\sum_{j=0}^{\infty}\binom{\rho(\beta-1)}{j}\lambda^j e^{-j\theta x^2}$$

$$\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)^{\rho}=\sum_{k=0}\binom{\rho}{k}(1-\lambda)^{(\rho-k)}(2\lambda)^k e^{-k\theta x^2}$$

Put $x^2 = t,\quad 2xdx = dt,$ as $x = 0,\ t = \infty$ and $x = \infty,\ t = 0$.

$$u(x) = \theta\beta^{\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}2^{k+\rho-1}\lambda^{(k+j)}(1-\lambda)^{(\rho-k)}\binom{\rho(\beta-1)}{i}\binom{\rho(\beta-1)}{j}\binom{\rho}{k}\int_0^{\infty}x^{\frac{\rho+1}{2}-1}e^{-(\rho+i+j+k)\theta t}dt$$

$$= \theta^{\left(\frac{\rho-1}{2}\right)}\beta^{\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}2^{k+\rho-1}\lambda^{(k+j)}(1-\lambda)^{(\rho-k)}\binom{\rho(\beta-1)}{i}\binom{\rho(\beta-1)}{j}\binom{\rho}{k}\frac{\Gamma\left(\frac{\rho+1}{2}\right)}{(\rho+i+j+k)^{\frac{(\rho+1)}{2}}}. \tag{6.3}$$

Putting the value of equation (6.3) in (6.2) we get the Renyi entropy of ETRD as follows:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left(\theta^{\left(\frac{\rho-1}{2}\right)}\beta^{\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}2^{k+\rho-1}\lambda^{(k+j)}(1-\lambda)^{(\rho-k)}\binom{\rho(\beta-1)}{i}\binom{\rho(\beta-1)}{j}\binom{\rho}{k}\frac{\Gamma\left(\frac{\rho+1}{2}\right)}{(\rho+i+j+k)^{\frac{(\rho+1)}{2}}}\right). \tag{6.4}$$

7. ORDER STATISTICS

Let $X_{(1)}, X_{(2)}, X_{(3)},\dots, X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2,\dots X_n$ of size n drawn from the ETRD having cdf and pdf given respectively by (2.3) and (2.4), then the pdf of rth order statistics of ETRD is given by:

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,. \qquad for\ 1\le r\le n \tag{7.1}$$

The pdf of the first order statistic $X_{(1)} = \min(X_1, X_2,\dots X_n)$ is given by:

$$f_1(x) = 2n\theta\beta x e^{-\theta x^2}\left\{1-\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{\beta}\right\}^{n-1}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{\beta-1}. \tag{7.2}$$

The pdf of the nth order statistic $X_{(n)} = \max(X_1, X_2,\dots X_n)$ is given by:

$$f_n(x) = 2n\theta\beta x e^{-\theta x^2}\left((1-\lambda)+2\lambda e^{-\theta x^2}\right)\left(1-e^{-\theta x^2}\right)\left(1+\lambda e^{-\theta x^2}\right)^{n\beta-1}. \tag{7.3}$$

8. Maximum Likelihood Estimation

We estimate the parameters of the proposed model using the method of MLE. The likelihood function is given by:

$$L(x) = (2\theta\beta)^n \prod_{i=1}^{n} \left\{ x_i e^{-\theta x_i^2} \left((1-\lambda) + 2\lambda e^{-\theta x_i^2}\right) \left(\left(1 - e^{-\theta x_i^2}\right)\left(1 + \lambda e^{-\theta x_i^2}\right)\right)^{\beta-1} \right\} \tag{8.1}$$

$$\therefore\ l = \log L(x) = n\log 2 + n\log\theta + n\log\beta + \sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\log\left((1-\lambda) + 2\lambda e^{-\theta x_i^2}\right) \tag{8.2}$$

$$+ (\beta-1)\sum_{i=1}^{n}\log\left(1 - e^{-\theta x_i^2}\right) + (\beta-1)\sum_{i=1}^{n}\log\left(1 + \lambda e^{-\theta x_i^2}\right)$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(1 - e^{-\theta x_i^2}\right) + \sum_{i=1}^{n}\log\left(1 + \lambda e^{-\theta x_i^2}\right). \tag{8.3}$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{-1 + 2e^{-\theta x_i^2}}{\left((1-\lambda) + 2\lambda e^{-\theta x_i^2}\right)} + (\beta-1)\sum_{i=1}^{n}\frac{2e^{-\theta x_i^2}}{\left(1 + \lambda e^{-\theta x_i^2}\right)}. \tag{8.4}$$

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}\frac{2\lambda x_i^2 e^{-\theta x_i^2}}{\left((1-\lambda) + 2\lambda e^{-\theta x_i^2}\right)} - (\beta-1)\sum_{i=1}^{n}\frac{x_i^2 e^{-\theta x_i^2}}{\left(1 - e^{-\theta x_i^2}\right)} - (\beta-1)\sum_{i=1}^{n}\frac{\lambda x_i^2 e^{-\theta x_i^2}}{\left(1 + \lambda e^{-\theta x_i^2}\right)}. \tag{8.5}$$

Now, solving the resulting non-linear system of equations $\frac{\partial l}{\partial\beta} = 0, \frac{\partial l}{\partial\theta} = 0$ and $\frac{\partial l}{\partial\lambda} = 0$

provides the maximum likelihood estimate of the parameters β, θ and λ respectively. Moreover, all the second order derivatives exist. Thus we have the3 x 3 inverse dispersion matrixes given by

$$\begin{pmatrix}\hat\beta\\ \hat\theta\\ \hat\lambda\end{pmatrix} \sim N\left[\begin{pmatrix}\beta\\ \theta\\ \lambda\end{pmatrix}, \begin{pmatrix}\hat V_{\beta\beta} & \hat V_{\beta\theta} & \hat V_{\beta\lambda}\\ \hat V_{\theta\beta} & \hat V_{\theta\theta} & \hat V_{\theta\lambda}\\ \hat V_{\lambda\beta} & \hat V_{\lambda\theta} & \hat V_{\lambda\lambda}\end{pmatrix}\right]$$

$$V^{-1} = -E\begin{pmatrix}V_{\beta\beta} & V_{\beta\theta} & V_{\beta\lambda}\\ V_{\gamma\beta} & V_{\theta\theta} & V_{\theta\lambda}\\ V_{\lambda\beta} & V_{\lambda\theta} & V_{\lambda\lambda}\end{pmatrix}, \tag{8.6}$$

where $V_{\alpha\alpha} = \frac{\partial^2 l}{\partial\beta^2}, V_{\gamma\gamma} = \frac{\partial^2 l}{\partial\theta^2}, V_{\lambda\lambda} = \frac{\partial^2 l}{\partial\lambda^2}$

and $V_{\theta\beta} = V_{\beta\theta} = \frac{\partial^2 l}{\partial\beta\,\partial\theta}, V_{\beta\lambda} = V_{\lambda\beta} = \frac{\partial^2 l}{\partial\beta\,\partial\lambda}, V_{\theta\lambda} = V_{\lambda\theta} = \frac{\partial^2 l}{\partial\theta\,\partial\lambda}$.

The solution of the above inverse dispersion matrix will yield the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\beta}, \hat{\theta}, \hat{\lambda}$. Hence, the approximate 100(1-α) % confidence intervals for β, θ, λ are given respectively by

$$\hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\beta\beta}}, \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\theta\theta}}, \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\lambda\lambda}}$$

where $Z_{\frac{\alpha}{2}}$ is the α^{-th} percentiles of the standard normal distribution.

9. Applications and Simulation Study

Here, we consider both a real life and simulated data sets to compare the flexibility of the ETR distribution over the existing sub models. The real data set is a subset of the data reported by Bekker et al. [1], which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone.

The simulated data sets of sizes 25, 50 and 100 have been generated from ETR distribution using the inverse CDF method with parameters (θ, λ, β) = (0.3, 0.3, and 0.2). The analysis involved in this study has been performed with the help of R software and the performances for the different sub models are presented in the tables given below:

Table 1: Distribution Performance with respect to simulated data sets

n	**Distribution**	θ	λ	β	**Log-likelihood**	**AIC**	**BIC**
25	ETRD	0.2854354 (0.21778624)	0.17840550 (1.17731203)	0.23117927 (0.05330543)	-10.68228	27.36457	31.02119
	TRD	0.7957615 (0.1957473)	0.6384628 (0.2226062)	–	-43.83593	91.67186	94.10961
	RD	0.9656273 (0.1931253)	–	–	-47.02985	96.05971	97.27858
	ETSRD	_	-0.5712586 (0.38564295)	0.25108595 (0.06117106)	-16.28081	36.56161	38.99936
	ESRD	0.2915334 (0.0583060)	–	–	-17.10664	36.21328	37.43216
	TSRD	_	0.5519817 (0.2071306)	–	-44.31726	90.63453	91.8534
50	ETRD	0.32041326 (0.21058372)	0.61850470 (0.84261497)	0.24188568 (0.03808559)	-14.05814	34.11628	39.85235
	TRD	1.0682605 (0.1921543)	0.6880692 (0.1680017)	–	-76.05998	156.12	159.944
	RD	1.3812709 (0.1953411)	–	–	-81.96145	165.9229	167.8349
	ETSRD	_	-0.76991238	0.22161842	-	38.20977	42.03381

			(0.28172753)	(0.04584189)	17.10488		
	ESRD			0.27926205 (0.03949311)	- 19.04657	40.09315	42.00517
	TSRD	_	0.7190950 (0.1398676)	_	- 76.12579	154.2516	156.1636
100	ETRD	0.3250869 (0.15547395)	0.80678763 (0.53612244)	0.24822879 (0.02822033)	- 24.28131	54.56262	62.37813
	TRD	1.2545354 (0.1644369)	0.6394561 (0.1269644)	_	-140.794	285.5879	290.7983
	RD	1.5837690 (0.1583768)	_	_	- 151.3273	304.6546	307.2598
	ETSRD	_	-0.5273861 (0.23884204)	0.2440504 (0.03136916)	- 27.99245	59.98491	65.19525
	ESRD			0.27931783 (0.02793143)	- 29.97287	61.94573	64.5509
	TSRD	_	0.75695959 (0.09720453)	_	- 142.1505	286.3011	288.9063

Table 2: Distribution Performance with respect to chemotherapy data

Distribution	θ	λ	β	Log-likelihood	AIC	BIC
ETRD	0.14353065 (0.0501614)	0.16638809 (0.49633472)	0.40243642 (0.07347305)	-58.63927	123.2785	128.6985
TRD	0.2516189 (0.0473479)	0.5225830 (0.19332706)	_	-74.69783	153.3957	157.009
RD	0.30130088 (0.0449149)	_	_	-77.91663	157.8333	159.6399
ETSRD	_	-0.4871733 (0.2713967)	0.6390210 (0.1293740)	-125.1827	254.3654	257.9787
ESRD	0.7856918 (0.1171238)	_	_	-126.8645	255.729	257.5357
TSRD	_	-0.0990113 (0.1883941)	_	-128.1472	258.2944	260.101

We noticed from Table 1 and 2 that ETR model gives the highest log-likelihood value or the lowest AIC and BIC values as compared to its different sub models. So, we conclude that the ETR distribution shows to a better fit than the TRD, RD, ETSRD, ESRD and TSR distribution and therefore could be chosen as the best model.

10. CONCLUSION

In this research paper, we proposed a new three-parameter probability model known as Exponentiated Transmuted Rayleigh Distribution (ETRD), which is an extension of the Rayleigh distribution. By considering both the simulated as well as real life data sets, we proved its applicability in comparison to other existing models like one Parameter Rayleigh Distribution, Exponentiated Rayleigh Distribution, Exponentiated Transmuted Standard Rayleigh Distribution, Transmuted Standard Rayleigh Distribution and Standard Rayleigh distribution. Thus, we conclude that our model is better as compared to other models mentioned above.

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