MODELLING ANOMALOUS TRANSPORT WITH EXTERNAL FIELD BY INTEGRATED SUBORDINATED BROWNIAN MOTION

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ABSTRACT

We consider a new type of anomalous transport in the particular type of continuous time random walk (CTRW) in the stochastic process where the particle have interaction with its environment and may behave unexpectedly more random or more stable. The process that was utilized in this study uses the integrated Brownian motion subordinated by an inverse α-stable subordinator. The proposed new process has external field which causes the two most probable points to have another point between them and we termed it as anomalous jump. We compute the time-scale for the mean squared displacement of the usual continuous time random walk (CTRW) and the anomalous jump. Furthermore, as expected, the time scale for the anomalous jump exhibits an interaction with its environment when there is an imposed memory kernel function. Lastly, using the time scale obtained for the MSD of the anomalous jump, we have generalized the external drift force field on the coupled Langevin equation obtained by Fogedby.

KEYWORDS

CTRW, Anomalous diffusions, Stochastic Processes, Subordinated Processes, Integrated Brownian Motion, Subordinator, Coupled Langevin Equation, Caputo Fractional Derivative.

1. INTRODUCTION

The concept of the continuous time random walk was first introduced by Montroll-Weisscher [1] and has been utilized over the past decades to model diffusions. Continuous time random walk(CTRW) can be considered as an evolution of a random walker's position and within this framework, it can be used to model anomalous behavior which can be characterized as anomalous diffusions. Over the past years, processes exhibiting anomalous increasingly attracted many attentions [2]. Such process can be used to study the particles in complex environments. In fact, there are alternative approaches [3] to these kinds of processes and one of which is the Focker-Plank equation that contains non-local fractional derivatives and the CTRW processes [4][5][6] which have been proposed to described the analysis on microscopic properties of anomalous diffusion processes. In the continuum limit, the continuous descriptions of CTRWs have been considered by Foged by [7] in 1994, which is a set of stochastic differential equations

$$\frac{dx}{ds} = F(x) + \eta(s)$$

$$\frac{dt}{ds} = \zeta(s) \tag{1}$$

where η(s) is a Gaussian noise with $\langle \eta(s) \rangle = 0$, $\langle(\eta(s)\eta(\tau)\rangle = \delta(s-\tau)$ and ζ(s) is a white α-stable L`evy noise. As can be seen in (1), the random walk x(t) = X(s(t)) is parametrized in terms of continuous random variable s, which is subjected to a random change of physical time t. In the absence of the external fied F(x) analytical expressions for correlation functions could be derivedby the application of the inverse Fourier and Laplace Transforms. If F(x) = −γx, we are dealing with the Ornstein-Uhlenbeck process [8].

The aim of this study is to make a model for CTRW which is associated with an anomalous jump and to describe its connection to equation (1) through the distribution of the time-scale of the MSD corresponding to the anomalous jumps.

2. MODEL

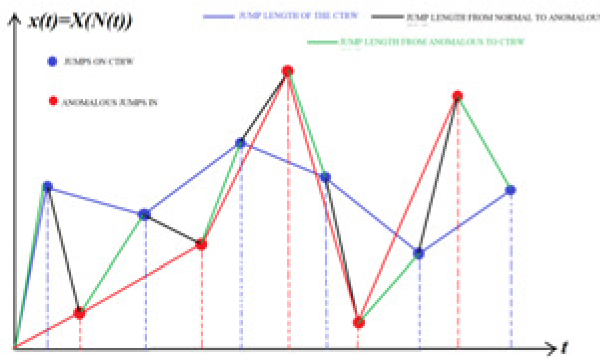

Figure 1: Schematic representation of the CTRW (Blue) and the anomalous jumps (Red)

In figure 1, let T_i and T_k be a non-negative series of independent identically distributed (IID) random variables denoting the waiting times between the jump on CTRW (Blue line in the Figure), and the anomalous jump (Black line in the figure), respectively. We set t(0) = 0 and t(n) $=\sum_{i=1}^{n} T_i + \sum_{i=1}^{n-1} T_k$ to denote the total time after nth jump including the anomalous one. Let L_i and L_k be the series of (IID) jump length of the CTRW (blue) and the anomalous jump (black). If we set X(0) = 0 and let X(n) $=\sum_{i=1}^{n} L_i + \sum_{i=1}^{n-1} L_k$ to denote the final position of the random walker, where $N(t) = max(n \geq 1 : t(n) \leq t)$ is the number of jump up to time t including the anomalous jump. In terms of subordinated process we can write position of the random walker as

$$x(t) = X\big(N(t)\big) = \sum_{i=1}^{N(t)} L_i + \sum_{i=1}^{N(t)-1} L_k \tag{2}$$

In the continuous limit, we can write the position of the CTRW with anomalous jump as

$$\frac{dX}{ds} = \int_0^s \chi(s')ds' + \int_0^s \chi(s'-1)ds' \tag{3}$$

$$t(s) = \int_0^s \zeta(s')ds' \tag{4}$$

where equation (2) has an equivalent representation in the form of

$$x(t) = X(s_\alpha(t)) \tag{5}$$

and the parent process X(s) takes the form

$$X(s) = \int_0^s B(s')ds' + \int_0^s B(s'-1)ds' \tag{6}$$

where B is the standard Brownian motion and the inverse subordinator $s\alpha(t)$ is defined by [3]

$$s_\alpha(t) = inf[\, s > 0 : \; t(s) \geq t] \tag{7}$$

or

$$s_\alpha(t) = sup[\, s > 0 : \; t(s) \leq t] \tag{8}$$

and equation (4) is an α-stable L`evy motion with characteristic form

$$\langle e^{-ut(s)} \rangle = e^{-u^\alpha s}, \quad 0 < \alpha \quad \leq 1 \tag{9}$$

3. RESULTS AND DISCUSSIONS

In order to calculate the probability distribution of the random walker for both CTRW and anomalous jump with physical time t, eliminating the variable s is required (see [7]). With this propostion, we define the Probability Density Functions $\Phi(x,t)$, $\psi(x,s)$, and $\Psi(s,t)$. In terms of the subordination process, we have

$$\Phi(x,t) = \int_0^\infty \psi(x,s)\, \Psi(s,t) ds \; (10)$$

For the first moment of the parent process, [5] we have

$$\langle X(s) \rangle = \langle \int_0^s B(s')ds' \rangle + \langle \int_0^s B(s'-1)ds' \rangle = 0 \tag{11}$$

and the second moment is

$$\langle X^2(s) \rangle = \langle \int_0^s B(s')ds' \cdot \int_0^s B(s'')ds'' \rangle$$

$$+ \langle \int_0^s B(s')ds' \cdot \int_0^s B(s'')ds'' \rangle$$

$$= \frac{2}{3}s^3 - s^2 \tag{12}$$

For the first moment of the subordinated process, we have

$$\langle x(t)\rangle = \int_{-\infty}^{+\infty} x\,\Phi(x,t)dx = \int_0^{\infty} dx \int_0^{\infty} x\psi(x,s)\Psi(s,t)ds = \int_0^{\infty} \langle X(s)\rangle \Psi(s,t)ds = 0 \quad (13)$$

$$\Rightarrow \langle x^2(t)\rangle = \int_{-\infty}^{+\infty} x^2\,\Phi(x,t)dx$$

$$= \int_0^{\infty} dx \int_0^{\infty} x^2\psi(x,s)\Psi(s,t)ds = \int_0^{\infty} \langle X^2(s)\rangle \Psi(s,t)ds$$

$$= \int_0^{\infty} \frac{2}{3}s^3\,\Psi(s,t)ds - \int_0^{\infty} s^2\,\Psi(s,t)ds \quad (14)$$

Hence, the MSD for the subordinated process takes the form

$$\langle(\Delta x)^2(t)\rangle = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = \int_0^{\infty} \frac{2}{3}s^3\,\Psi(s,t)ds - \int_0^{\infty} s^2\,\Psi(s,t)ds \quad (15)$$

From the relations shown in equation (7) and (8), we have the equality of the probability distribution

$$P(s_\alpha \le s) = 1 - P(t(s) \le t) \quad (16)$$

which then allows us to write the PDF $\Psi(s,t)$ in terms of$\Psi^*(t,s)$ as

$$\Psi(s,t) = -\frac{\partial}{\partial s}\int_0^{t} \Psi^*(t',s)ds \quad (17)$$

Taking its Laplace Transform yields

$$\widetilde{\Psi}(s,u) = -\frac{\partial}{\partial s}\frac{1}{u}\widetilde{\Psi}^*(u,s) = u^{\alpha-1}e^{-u^\alpha s}, 0 < \alpha \le 1 \quad (18)$$

We have the MSD for the subordinated process in Laplace Space

$$\langle\left(\widetilde{\Delta x}\right)^2(u)\rangle = \int_0^{\infty} \frac{2}{3}s^3u^{\alpha-1}e^{-u^\alpha s}ds - \int_0^{\infty} s^2u^{\alpha-1}e^{-u^\alpha s}ds$$

$$= \frac{4}{u^{2\alpha+2}} - \frac{2}{u^{\alpha+2}} \quad (19)$$

Finally, we obtained the MSD for the subordinated process in terms of its physical time-scale

$$\langle(\Delta x)^2(t)\rangle = 4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{20}$$

The first term of equation (20) corresponds to the physical time scale of the normal jump [5], while the second term corresponds to the physical time scale of the anomalous jump. Hence, the total expression of the obtained MSD in Eq. (20) is the MSD time scale for the whole process with the inclusion of the anomalous jump to go to the expected normal jump. This would suggest that the anomalous jump can be triggered by the external field, and it is possible to be characterized. The next step is to change the expression (α +2) $\rightarrow \lambda$, then the 2nd term of the right side of the eq. (20) will become

$$\langle(\Delta x)^2(t)\rangle = \frac{t^{\lambda-1}}{\Gamma(\lambda)} \tag{21}$$

Now, consider a memory function *R(t)* which can be represented as Brownian Process and a Fractional integrand of the form [10]

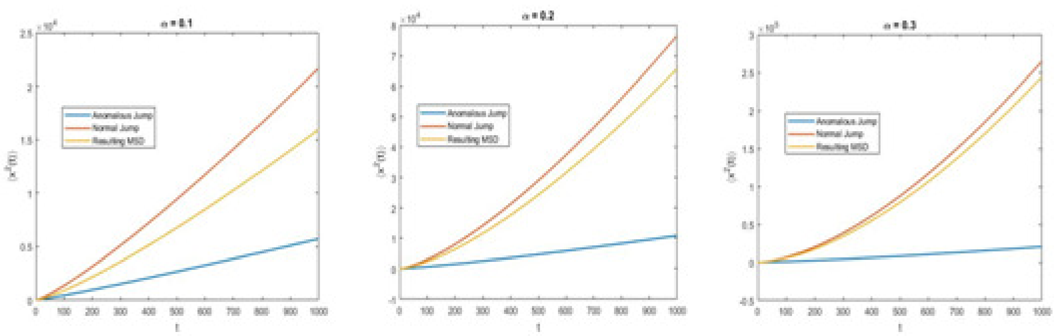

Figure 2**:** Graphical illustration of equation (20) where it also follow a power-law behavior. It shows that for small α, deviation is very clear between the processes and this suggests that for larger α, the resulting mean squared displacement is restored (e.g if $t \rightarrow \infty$, corresponding MSD for the normal jump is recovered)

$$I_{t_o}^{\lambda} R(t) = \frac{1}{\Gamma(\lambda)} \int_{t_o}^{t} (t-t')^{\lambda-1} R(t') dt' \tag{22}$$

where t_o is the starting point for the function R(t) to occur and (21) is the known Reimann-Louiville fractional integration. In order for (22) to be legitimate, we should consider the possibility to change $\lambda \rightarrow -\lambda$. Let λ be any positive real integer in the interval $(m-1, m)$ with $m \in \aleph$ then R(t) can be considered as a well-behaved function. Introducing a fractional derivative of R(t) of order λ

$$D_{t_o}^{\lambda} R(t) = \frac{1}{\Gamma(m-\lambda)} \frac{d^m}{dt^m} \int_{t_o}^{t} (t-t')^{\lambda-1} R(t') dt' \tag{23}$$

where we can write (23) in terms of Caputo Fractional time derivative

$$ {}_{C}D^{\lambda}_{t_o}R(t) = \frac{1}{\Gamma(m-\lambda)}\int_{t_o}^{t}(t-t')^{m-\lambda-1}R^{m}(t')dt' \tag{24} $$

Since the memory function R(t) describes the movement of the anomalous jump, it vanishes at t^{-}_{o}. Then, it is convenient to define the Gel'fand-Shilvo distribution

$$ \phi_o(t_o) = \frac{t^{\sigma-1}}{\Gamma(\sigma)}H_o(t_o) \tag{25} $$

where the $H(t_o)$ is the unit step Heaviside function. Now, suppose (24) is legitimate under its m−derivative integral. Then, we can write (23) as

$$ D^{\lambda}_{t_o}R(t) = \ {}_{C}D^{\lambda}_{t_o}R(t) + \sum_{l=0}^{m-1}R^{(l)}(t_o^{+})\phi_{l-\lambda+1}(t) \tag{26} $$

and the first term of (26) can be written in Laplace Space as

$$ L\left(\ {}_{C}D^{\lambda}_{t_o}R(t)\right) = q^{\lambda}R(q) - \sum_{l=0}^{m-1}q^{\lambda-l-1}R^{(l)}(t_o^{+}) \tag{27} $$

For $\lambda = 1/2, m = 1$, notice that(24) takes the form

$$ {}_{C}D^{\lambda}_{t_o}R(t) = \frac{1}{\Gamma(1/2)}\int_{t_o}^{t}(t-t')^{-1/2}\frac{dR(t')}{dt'}dt' \tag{28} $$

is the known Basset force for $t > t_o$. And for $m = 2$and $\lambda = 1$,it takes the form which satisfies the friction force due to the plasma fluids. Thus, we can say that the anomalous jump in CTRW can be triggered by a fractional force of the form (24). Therefore, the coupled Langevin equation (1) can be generalized in terms of a fractional force and maybe written in the for

$$ \frac{dx}{ds} = F(x) + \eta(s) = \ {}_{C}D^{\lambda}_{t_o}x(t) + \eta(s) $$

which is now considered by many authors for modeling fractional brownian motion in complex environment given by proper conditions and parameter.

4. CONCLUSION

It is found in this study that considering a random jump between two expected or more probable points exhibits another time scale which is less than the normal time scale for anomalous transport given in the first term of equation (20). In contrast to other works [2],[4],[5],[7], where we considered new behavior, it is found that the corresponding time scale for the anomalous transport with external field is given by equation (20) and further details for the time scale corresponding to the external field was elaborated upon considering a memory function, thus resulting to equation (21) and (23). In addition, the external field was found to be characterized by the Caputo fractional derivative with proper parameters.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publication of this paper.

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