ON THE PROJECTIVE CHANGE OF SECOND APPROXIMATE MATSUMOTO METRIC WITH CERTAIN (α, β)-METRIC

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ABSTRACT

In the present paper, we find equations to characterize the projective changes between two important (α, β) *-metric which are* $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ *(second approximate Matsumoto metric) and* $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ *(Randers metric) and also between second approximate Matsumoto metric and* $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ *(Kropina metric), on a manifold with dimension* $n \geq 3$ *where* α *and* $\tilde{\alpha}$ *are two Riemannian metrics,* β *and* $\tilde{\beta}$ *are two non-zero one forms. Moreover we consider this projective change when F has some special curvature properties.*

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KEYWORDS:

Finsler metric, (α, β) *-metric, Kropina metric, Projective Change, Douglas metric and S-curvature.*

1. INTRODUCTION

The Projective changes between two Finsler spaces have been researched and thought through by many geometers (see [4], [14], [15], [17]). It's been defined that two Finsler metrics on a smooth manifold M are considered to be Projectively equivalent in case they consists of the same geodesics as point sets and their geodesic coefficients is determined by the relation

$$G^i = \tilde{G}^i + P(x, y) y^i .$$

where $P(x, y)$ is supposed to be a scalar function on $TM \setminus \{0\}$ with $P(x, \lambda y) = \lambda P(x, y)$ and the two Riemmanian metrics are considered to be Projectively equivalent on the condition that their spray coefficients are related by

$$G^i_\alpha = \tilde{G}^i_\alpha + \tau_{x^k} y^k y^i ,$$

here $\tau = \tau(x)$ represents a scalar function on the manifold M . Local coordinates in the tangent bundle TM is denoted by (x^i, y^j) .

In Finsler geometry (α, β) -metric is notified as a substantial and significant class of Finsler metrics. It can be depicted in the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, ,where α is the Riemannian metric, β represents one form and ϕ denotes the positive C^∞ function on the domain of definition. Exceptionally, when $\phi = \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called Kropina metric. L. Berwald was the first one to introduce Kropina metric in connection with a two-dimensional Finsler space with rectilinear extremals and was studied upon by V. K. Kropina [7]. Whereas, Randers metric is regular Finsler metric, on the other hand Kropina metric is non-regular Finsler metric. Kropina metric is considered to be one of the significant and elementary Finsler metric with abundance of interesting and useful applications in physics, irreversible thermodynamics, dissipative mechanics and electron optics with a magnetic field ([6], [16]). Besides this, it has uses in applications related to control theory, relativistic field theory, developmental biology and evolution.

Rapsack's paper [13] has provided us a very important and necessary result related with the projective change, which deals with the necessity and sufficiency of Projective change. H. Park and Y. Lee , in 1984 [11] studied and put limelight on the projective change between a Finsler space with (α, β) -metric and the associated Riemmanian metric. In similar way, numerous papers have been devoted on the topic 'Projective change'. As we have more examples in its context like Projective change between Finsler spaces with (α, β) -metric, studied by S. Bacso and M. Matsumoto [2]. A class of Projectively at metrics with constant flag curvature has been researched upon by Z. Shen and Civi Yildrim in [15]. In 2009, N. Cui and Y. Shen [4] were the ones who did a deep study on projective change between Z. Shen square metric and a Randers metric. Recently in 2012, Jingjnong and Xinyue Cheng carried further the topic of projective changes between (α, β) -metric dealing with Randers metric and Kropina metric.

2. PRELIMINARIES

The geodesics of F are defined by a system of 2nd order differential equations as follows,

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $F = F(x, y)$ represents a Finsler metric.

A Finsler metric on a manifold M is a C^∞ -function $F : TM \to [0, \infty)$ satisfies the following properties:

1. Regularity: F is C^∞ on $TM \setminus \{0\}$;

2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

3. Strong convexity: The fundamental tensor $g_{ij}(x, y)$ is positive for all $(x, y) \in TM \setminus \{0\}$;

where $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$. The pair $(M, F) = F^n$ is called Finsler space. F is called the fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

$G^i = G^i(x, y)$ are called spray coefficients of F, given by

$$G^i = \frac{1}{4} g^{il} \left[[F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right].$$

Let

$$D^i_{jkl} = \tilde{D}^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i_\alpha - \frac{1}{n+1} \frac{\partial G^m_\alpha}{\partial y^m} y^i + T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right)$$

$$= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \tag{2.1}$$

The tensor $D := D^i_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called the Douglas metric if the Douglas tensor vanishes.

It can be easily reviewed that the Douglas tensor is a projective invariant. In addition to this we have a vital fact which states that all Berwald metrics must be Douglas metrics.

For a (α, β)-metric,

$$F = \alpha \phi(s), s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij} y^i y^j}$ represents a Riemannian metric and $\beta = b_i(x) y^i$ denotes a one form with $\|\beta\| < b_0$. For $F = \alpha \phi\left(\frac{\beta}{\alpha}\right)$ to be a regular Finsler metric ([1], [3]), the function $\phi(s)$ has to be positive C^∞ function on an open interval $(-b_0, b_0)$ satisfying,

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \le b \le b_0.$$

One knows that Randers metric is regular on the other hand Kropina metric is not regular, still the relation

$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$ is completely valid for $|s| > 0$.

The geodesic coefficients of F and α are depicted in the form $G^i(x,y)$ and $G^i_\alpha(x,y)$, respectively and the covariant derivative of β with respect to α is denoted by $\nabla\beta = b_{i|j}dx^i \otimes dx^j$. Thus we have

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), r_i := r_{ij}b^j$$

and put $r_{00} := r_{ij}y^i y^j, r_0 := r_j y^j, s_{l0} := s_{li}y^i, s_0 := b^l s_{l0}$, etc. Importantly the geodesic coefficient $G^i(x,y)$ of F is defined by, [11]

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \Theta(2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + \psi(-2\alpha Q s_0 + r_{00})b^i \ , \tag{2.2}$$

where $s^i_j := a^{ik}s_{kj}$ and

$$Q = \frac{\phi^{'}}{\phi - s\phi^{'}},$$

$$\Theta = \frac{\phi\phi^{'} - s(\phi\phi^{''} + \phi^{'}\phi^{'})}{2\phi[(\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}]},$$

$$\psi = \frac{\phi^{''}}{2[(\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}]}, \tag{2.3}$$

The (α,β)-metrics of Douglas type have been illustrated in [8].

Further to find the desired results, firstly we calculate the douglas tensor of (α,β)-metrics.
Since

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \psi(-2\alpha Q s_0 + r_{00})b^i .$$

Clearly, the sprays G^i and $\tilde{G}^i$ are projective invariant providing the same Douglas tensor. Let

$$T^i = \alpha Q s^i_0 + \psi(-2\alpha Q s_0 + r_{00})b^i . \tag{2.4}$$

Then $\tilde{G}^i = G^i_\alpha + T^i$.

From (2.3), we get

$$T^m_{y^m} = \frac{\partial T^m}{\partial y^m} = Q^{'}s_0 + 2\psi[r_0 - Qss_0 - Q^{'}(b^2 - s^2)s_0] - \psi^{'}\alpha^{-1}(b^2 - s^2)[2\alpha Q s_0 - r_{00}]. \tag{2.5}$$

Now, if the metrics F and $\tilde{F}$ consists of the same Douglas tensor, i.e. $D^i_{jkl} = \tilde{D}^i_{jkl}$, , by definition of Douglas tensor and (2.5), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i \right) = 0$$

Thus we have a class of scalar functions, given by $H^i_{jk} := H^i_{jk}(x)$, such that

$$T^i - \tilde{T}^i - \frac{1}{n+1}(T^m_{y^m} - \tilde{T}^m_{y^m}) y^i = H^i_{00}, \qquad (2.6)$$

where $H^i_{00} := H^i_{jk} y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations (2.3) and (2.5) respectively.

3. PROJECTIVE CHANGE BETWEEN TWO (α, β) -METRICS.

For a Finsler space $F^n = (M, F)$, the metric $F = F(x, y)$ is considered as a Finsler metric provided $\|\beta\| < b_0$ and their geodesic coefficients are given by (2.1) and (2.2). One can easily obtain the following:

(a.) For **Second approximate Matsumoto metric** $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$, we have

$$Q = \frac{1 + 2s + 3s^2}{1 - s^2 - 2s^3}; \quad \Theta = \frac{1 - 6s^2 - 12s^3 - 15s^4 - 12s^5}{2\{1 - 3s^2 - 8s^3 + 2b^2(1+3s)\}(1 + s + s^2)};$$

$$\psi = \frac{1 + 3s}{1 - 3s^2 - 8s^3 + 2b^2(1+3s)}. \qquad (3.1)$$

(b.) For **Randers metric** $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$, we have

$$\tilde{Q} = 1; \qquad \tilde{\Theta} = \frac{1}{2(1+s)}; \qquad \tilde{\psi} = 0. \qquad (3.2)$$

(c.) For **Kropina metric** $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$, we have

$$\tilde{Q} = -\frac{1}{2s}; \qquad \tilde{\Theta} = -\frac{s}{\tilde{b}^2}; \qquad \tilde{\psi} = \frac{1}{2\tilde{b}^2}. \qquad (3.3)$$

Now we discuss the projective change between two (α, β) -metrics,

3.1 Projective change between $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ (2nd approximate Matsumoto metric) and $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ (Randers metric).

Since the Douglas tensor is a Projective invariant, we have our respective propositions,

Proposition 3.1. Let us consider an (α,β)-metric given by $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ and a Randers metric defined as $\tilde{F}=\tilde{\alpha}+\tilde{\beta}$ on an n-dimensional manifold $(n\geq 3)$, where α and $\tilde{\alpha}$ denotes distinguish Riemannian metrics whereas β and $\tilde{\beta}$ denote non-zero 1-forms. The metrics F and $\tilde{F}$ are Projectively equivalent provided both are Douglas metrics and their spray coefficients are related by following equations

$$G^i_\alpha=\tilde{G}^i_{\tilde{\alpha}}+Py^i, \tag{3.4}$$

$$b_{i|j}=2\tau[(1+2\kappa b^2)a_{ij}-3\kappa b_i b_j], \tag{3.5}$$

where $P=P(x,y)$ is a scalar function on $TM\setminus\{0\}$ and $b_{i|j}$, represents covariant derivatives of β with respect to α.

Proof: Since the sufficiency is obvious we need to prove the necessity. If F and $\tilde{F}$ consists of the same Douglas tensor on M, then the equation (2.6) is valid. On substituting (3.1) and (3.2) into (2.6), we get

$$\begin{aligned}&[(A^i_1\alpha^{13}+A^i_2\alpha^{12}+A^i_3\alpha^{11}+A^i_4\alpha^{10}+A^i_5\alpha^9+A^i_6\alpha^8+A^i_7\alpha^7+A^i_8\alpha^6+A^i_9\alpha^5+A^i_{10}\alpha^4+A^i_{11}\alpha^3\\&+A^i_{12}\alpha^2+A^i_{13}\alpha+A^i_{14})/(P_1\alpha^{12}+P_2\alpha^{11}+P_3\alpha^{10}+P_4\alpha^9+P_5\alpha^8+P_6\alpha^7+P_7\alpha^6+P_8\alpha^5\\&+P_9\alpha^4+P_{10}\alpha^3+P_{11}\alpha^2+P_{12}\alpha+P_{13})]-\tilde{\alpha}\tilde{s}^i_0=H^i_{00}.\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}A^i_1&=(1+2b^2)^2s^i_0,\\A^i_2&=(1+2b^2)^2(2s_0(1-\lambda)y^i+r_{00}b^i)+6\beta b^2s^i_0-(1+2b^2)(r_0-b^2s_0)\lambda y^i,\\A^i_3&=-4\beta^2(1-b^2+b^4)s^i_0-\beta b^2\{3(1+2b^2)-16\}s_0\lambda y^i-12\beta b^2r_0\lambda y^i+3[\beta b^i\{(1+2b^2)^2+2b^2\}\\&\quad+b^2\lambda y^i]r_{00},\\A^i_4&=\beta(-32\beta^2s^i_0+\beta s_0[2\beta\{15(3+4b^2)+24b^4\}]b^i+[\{2\beta(3+4b^2)-226b^4+3(1+2b^2)\}\lambda y^i]\\&\quad+\beta r_0\{2\beta(5+4b^2)+3(1+2b^2)\}\lambda y^i-\beta r_{00}[\{5(1-2b^2)+8b^4\}b^i+6b^2\lambda y^i]),\end{aligned}$$

$$A_5^i = \beta^2(-4\beta^2(59b^2+24)s_0^i + s_0 b^2[172\beta^2 b^i + \{266-\beta(67+232b^2)\}]\lambda y^i - 5r_0(5-8\beta b^2)\lambda y^i$$

$$- r_{00}[\beta\{(17-38b^2)b^i + 32b^4\} - 3(1+10b^2)\lambda y^i]),$$

$$A_6^i = \beta^3(2\beta^2(13-119b^2)^2 s_0^i + s_0[\beta(6+b^2+b^4)b^i + [4b^2\{\beta(172+9b^2)-57b^2\}+3+329\beta$$

$$+18b^2]]\lambda y^i + 2r_0\{2\beta(2+11b^2)+6(1+4b^2)\}\lambda y^i - r_{00}[\{\beta(4+11b^2)+29\}b^i$$

$$-6(1-4b^2)\lambda y^i]),$$

$$A_7^i = \beta^4(42\beta^2(6+\beta)s_0^i + s_0[4\beta^2(44+123b^2+484b^4+100\beta)b^i - \{36b^2(1+5\beta b^2)$$

$$-\beta(493+936b^2)+12\beta^3(6+19b^2)\}\lambda y^i] - 2r_0\beta\{22b^2+45)-6\}\lambda y^i + r_{00}[\beta\{3\beta(1+8b^2)$$

$$+2(9+11b^2)\}b^i - (21-75b^2)\lambda y^i]),$$

$$A_8^i = \beta^5(346\beta^2 s_0^i + s_0[2\beta^2(223+248b^2)b^i - \{\beta(1156-b^2(54b^2-163))\} - 9b^2\{(1+2b^2+75)$$

$$+12\}\lambda y^i] - 2r_0\{\beta(-91+19b^2)+3b^2\}\lambda y^i + 2r_{00}\{\beta(34+47b^2+52b^4)+(15+105b^2)\}\lambda y^i),$$

$$A_9^i = \beta^6(\beta^2(343+384b^4)s_0^i + s_0[\beta^2\{24(1+39b^2)b^i - \beta\{(-324+59b^2)+12(1+2b^2)\}\lambda y^i\}]$$

$$-3r_0\beta\{(13+24b^2)+4(1+2b^2)\}\lambda y^i + r_{00}\beta\{125+4b^2(51+24b^2)b^i - 3(9-41b^2)\lambda y^i\}),$$

$$A_{10}^i = \beta^7(-102\beta^2 s_0^i + 2s_0[192\beta^2 b^i - \{9b^2(1+37\beta)-1171\beta\}\lambda y^i] + r_{00}\{12\beta(1+2b^2)b^i$$

$$-6(15+34b^2)\lambda y^i\}),$$

$$A_{11}^i = \beta^8(-400\beta^2 s_0^i + 2159\beta s_0\lambda y^i + 228\beta r_0\lambda y^i + r_{00}\{-80\beta b^i - 3(1+108b^2)\lambda y^i\}),$$

$$A_{12}^i = \beta^9(348\beta^2 s_0^i + 96\beta r_0\lambda y^i + 1032\beta s_0\lambda y^i + 6r_{00}\{16\beta b^i + (53-32b^2)\lambda y^i\}),$$

$$A_{13}^i = 420\beta^{10} r_{00}\lambda y^i,$$

$$A_{14}^i = -192\beta^{11} r_{00}\lambda y^i$$

(3.7)

and

$$P_1 = (1+2b^2)^2,$$

$$P_2 = 12\beta b^2(1+2b^2),$$

$$P_3 = -8\beta^2(1+b^2+b^4),$$

$$P_4 = -4\beta^3\{1+2b^2(5-8b^2)-9b^2\},$$

$$P_5 = 4\beta^4\{1-29b^2(1+b^2)+3\beta^2 b^4\},$$

$$P_6 = 4\beta^5\{3(1+2b^2)+b^2(45-4b^2)\},$$

$$P_7 = 2\beta^6\{(1+2b^2)(7+25b^2)+186b^2\},$$

$$P_8 = \beta^7\{33b^2+(1+2b^2)(23+6b^2)\},$$

$$P_9 = \beta^8(235-392b^2+24b^4),$$

$$P_{10} = 20\beta^9(7-26b^2),$$
$$P_{11} = -\beta^{10}\{82(1+2b^2)-28b^2\},$$
$$P_{12} = -352\beta^{11},$$
$$P_{13} = -128\beta^{12}, \tag{3.8}$$

and $\lambda = \dfrac{1}{n+1}$.

Thus (3.6) gives

$$A_1^i\alpha^{13} + A_2^i\alpha^{12} + A_3^i\alpha^{11} + A_4^i\alpha^{10} + A_5^i\alpha^9 + A_6^i\alpha^8 + A_7^i\alpha^7 + A_8^i\alpha^6 + A_9^i\alpha^5 + A_{10}^i\alpha^4 + A_{11}^i\alpha^3 + A_{12}^i\alpha^2 + A_{13}^i\alpha + A_{14}^i = (H_{00}^i + \tilde{\alpha}\tilde{s}_0^{\,i})(P_1\alpha^{12} + P_2\alpha^{11} + P_3\alpha^{10} + P_4\alpha^9 + P_5\alpha^8 + P_6\alpha^7 + P_7\alpha^6 + P_8\alpha^5 + P_9\alpha^4 + P_{10}\alpha^3 + P_{11}\alpha^2 + P_{12}\alpha + P_{13}).$$
(3.9)

Replacing y^i by $-y^i$ in (3.9), we get

$$-A_1^i\alpha^{13} + A_2^i\alpha^{12} - A_3^i\alpha^{11} + A_4^i\alpha^{10} - A_5^i\alpha^9 + A_6^i\alpha^8 - A_7^i\alpha^7 + A_8^i\alpha^6 - A_9^i\alpha^5 + A_{10}^i\alpha^4 - A_{11}^i\alpha^3 + A_{12}^i\alpha^2 - A_{13}^i\alpha + A_{14}^i = (H_{00}^i - \tilde{\alpha}\tilde{s}_0^{\,i})(P_1\alpha^{12} - P_2\alpha^{11} + P_3\alpha^{10} - P_4\alpha^9 + P_5\alpha^8 - P_6\alpha^7 + P_7\alpha^6 - P_8\alpha^5 + P_9\alpha^4 - P_{10}\alpha^3 + P_{11}\alpha^2 - P_{12}\alpha + P_{13}). \tag{3.10}$$

Subtracting (3.10) from (3.9), we get

$$A_1^i\alpha^{13} + A_3^i\alpha^{11} + A_5^i\alpha^9 + A_7^i\alpha^7 + A_9^i\alpha^5 + A_{11}^i\alpha^3 + A_{13}^i\alpha = (\tilde{\alpha}\tilde{s}_0^{\,i})(P_2\alpha^{11} + P_4\alpha^9 + P_6\alpha^7 + P_8\alpha^5 + P_{10}\alpha^3 + P_{12}\alpha + P_{13}. \tag{3.11}$$

Adding (3.9) and (3.10), we get

$$A_2^i\alpha^{12} + A_4^i\alpha^{10} + A_6^i\alpha^8 + A_8^i\alpha^6 + A_{10}^i\alpha^4 + + A_{12}^i\alpha^2 + A_{14}^i = (H_{00}^i)(P_1\alpha^{12} + P_3\alpha^{10} + P_5\alpha^8 + P_7\alpha^6 + P_9\alpha^4 + P_{11}\alpha^2 + P_{13}). \tag{3.12}$$

From (3.11) we can see that $P_{13}\tilde{\alpha}\tilde{s}_0^{\,i}$ has the factor α. Now we divide the proof in two cases.

Case I If $\tilde{\alpha} \neq \lambda(x)\alpha$, then $P_{13}\tilde{s}_0^{\,i} = -128\beta^{12}\tilde{s}_0^{\,i}$ has the factor α^2. Because β^{12} and α are relatively prime polynomials of y^i, then $\tilde{s}_0^{\,i} = 0$, , which implies $\tilde{\beta}$ is closed.

Case II If $\tilde{\alpha} = \lambda(x)\alpha$, then (3.11) reduces to

$$(A_1^i\alpha^{12} + A_3^i\alpha^{10} + A_5^i\alpha^8 + A_7^i\alpha^6 + A_9^i\alpha^4 + A_{11}^i\alpha^2 + A_{13}^i)\alpha = (P_2\alpha^{10} + P_4\alpha^8 + P_6\alpha^6 + P_8\alpha^4 + P_{10}\alpha^2 + P_{12})\alpha\lambda(x)\tilde{s}_0^{\,i} + \lambda(x)\tilde{s}_0^{\,i}P_{13}. \tag{3.13}$$

We observe that $\lambda(x)\tilde{s}_0^{\,i}P_{13} = -128\lambda(x)\beta^{12}\tilde{s}_0^{\,i}$ has the factor α. Since $\lambda(x) \neq 0$ then $\beta^{12}\tilde{s}_0^{\,i}$ has the factor α implying $\tilde{s}_0^{\,i} = 0,$, i.e $\tilde{\beta}$ is closed.

Apparently Randers metric $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ is a Douglas metric provided $\tilde{\beta}$ is closed. Thus we say $\tilde{F}$ is a Douglas metric and since F and $\tilde{F}$ are having alike Douglas tensor, hence they both are Douglas metrics. Thus proving proposition 3.1.

Now we are accessible to prove the following theorem,

Theorem 3.1. Let us consider an (α,β) -metric given by $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ and a Randers metric defined as $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ on an n-dimensional manifold $(n \geq 3)$, where α and $\tilde{\alpha}$ denotes distinguish Riemannian metrics whereas β and $\tilde{\beta}$ denote non-zero 1-forms. The metrics F and $\tilde{F}$ are Projectively equivalent provided both are Douglas metrics and their geodesic coefficients are related by following equations

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + Py^i, \tag{3.14}$$

where $P = P(x, y)$ is a scalar function on $TM \setminus \{0\}$.

Proof: Since F and $\tilde{F}$ are Projectively equivalen, they are having alike Douglas tensor implying that both are Douglas metrics. By [12], we know that (α,β) -metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Douglas metric if and only if

$$b_{i|j} = 0 \tag{3.15}$$

where $b_{i|j}$ represents the covariant derivatives of β with respect to α.
On substituting (3.15) and (3.1) into (2.1), we get

$$G^i = G^i_\alpha. \tag{3.16}$$

Since F is Projectively equivalent to $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ and $\tilde{\beta}$ is closed, F is Projectively equivalent to $\tilde{\alpha}$. Hence we have a scalar function $P = P(x, y)$ on $TM \setminus \{0\}$ such that

$$G^i = \tilde{G}^i_{\tilde{\alpha}} + Py^i, \tag{3.17}$$

From (3.16) and (3.17), we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + Py^i, \tag{3.18}$$

Thus proving the necessary part of the theorem. As F and $\tilde{F}$ are Projectively equivalent which completes the proof of theorem 3.1.

Now we will show the projective equivalence between second pair of metrics,

3.2 Projective change between $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ (2nd approximate Matsumoto metric) and Kropina metric $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$

First, we state the following

Lemma 3.2. *[8] Let* $F=\frac{\alpha^2}{\beta}$ *be a Kropina metric on an n-dimensional manifold M. Then*

(1) For $(n\geq 3)$*, Kropina metric* F *with* $b^2\neq 0$ *is a Douglas metric if and only if*

$$s_{ik}=\frac{1}{b^2}(b_i s_k - b_k s_i) \tag{3.19}$$

(2) For $n=2$, Kropina metric F is a Douglas metric.

Following from [8] and [9] and bringing Theorem 3.1 in use, we immediately obtain:

Proposition 3.1. Let us consider an (α,β)-metric given by $F=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ and a Randers metric defined as $\tilde{F}=\frac{\tilde{\alpha}^2}{\tilde{\beta}}$ on an n-dimensional manifold with $n\geq 3$, where α and $\tilde{\alpha}$ denotes distinguish Riemannian metrics whereas β and $\tilde{\beta}$ denote non-zero 1-forms. The metrics F and $\tilde{F}$ are Projectively equivalent provided following equations hold

$$G^i_\alpha=\tilde{G}^i_{\tilde{\alpha}}+\frac{1}{2\tilde{b}^2}(\tilde{\alpha}^2\tilde{s}^i+r_{00}\tilde{b}^i)+\theta y^i, \tag{3.20}$$

$$b_{i|j}=2\tau[(1+2\kappa b^2)a_{ij}-3\kappa b_i b_j], \tag{3.21}$$

where $b_{i|j}$, represents covariant derivatives of β with respect to α.

Proof. Here we need to prove the necessary part since the sufficiency of the proposition is obvious. Let us assume the metrics F and $\tilde{F}$ with the same Douglas tensor on an n-dimensional manifold M $(n\geq 3)$, then (2.6) is valid. On substituting (3.1) and (3.3) into (2.6), we get

$$[(A^i_1\alpha^{13}+A^i_2\alpha^{12}+A^i_3\alpha^{11}+A^i_4\alpha^{10}+A^i_5\alpha^9+A^i_6\alpha^8+A^i_7\alpha^7+A^i_8\alpha^6+A^i_9\alpha^5+A^i_{10}\alpha^4+A^i_{11}\alpha^3$$

$$+A_{12}^i\alpha^2 + A_{13}^i\alpha + A_{14}^i)/(P_1\alpha^{12} + P_2\alpha^{11} + P_3\alpha^{10} + P_4\alpha^9 + P_5\alpha^8 + P_6\alpha^7 + P_7\alpha^6 + P_8\alpha^5$$
$$+P_9\alpha^4 + P_{10}\alpha^3 + P_{11}\alpha^2 + P_{12}\alpha + P_{13})] + \frac{\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i}{2\tilde{b}^2\tilde{\beta}} = H_{00}^i. \quad (3.22)$$

where values of all the coefficients of different powers of α are stated in equations (3.7) and (3.8) and

$$\tilde{A}^i = \tilde{b}^2\tilde{s}_0^i - \tilde{b}^i\tilde{s}_0,$$
$$\tilde{B}^i = \tilde{\beta}[2\lambda y^i(\tilde{r}_0 + \tilde{s}_0) - \tilde{b}^i\tilde{r}_{00}],$$
$$\lambda = \frac{1}{n+1}.$$

Further, (3.22) is equivalent to

$$(A_1^i\alpha^{13} + A_2^i\alpha^{12} + A_3^i\alpha^{11} + A_4^i\alpha^{10} + A_5^i\alpha^9 + A_6^i\alpha^8 + A_7^i\alpha^7 + A_8^i\alpha^6 + A_9^i\alpha^5 + A_{10}^i\alpha^4 + A_{11}^i\alpha^3$$
$$+A_{12}^i\alpha^2 + A_{13}^i\alpha + A_{14}^i)(2\tilde{b}^2\tilde{\beta}) + (\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)(P_1\alpha^{12} + P_2\alpha^{11} + P_3\alpha^{10} + P_4\alpha^9 + P_5\alpha^8 + P_6\alpha^7 +$$
$$P_7\alpha^6 + P_8\alpha^5 + P_9\alpha^4 + P_{10}\alpha^3 + P_{11}\alpha^2 + P_{12}\alpha + P_{13}) = H_{00}^i(2\tilde{b}^2\tilde{\beta})(P_1\alpha^{12} + P_2\alpha^{11} + P_3\alpha^{10} + P_4\alpha^9$$
$$+P_5\alpha^8 + P_6\alpha^7 + P_7\alpha^6 + P_8\alpha^5 + P_9\alpha^4 + P_{10}\alpha^3 + P_{11}\alpha^2 + P_{12}\alpha + P_{13}).$$
(3.23)

Replacing y^i by $-y^i$ in (3.23), we get

$$(-A_1^i\alpha^{13} + A_2^i\alpha^{12} - A_3^i\alpha^{11} + A_4^i\alpha^{10} - A_5^i\alpha^9 + A_6^i\alpha^8 - A_7^i\alpha^7 + A_8^i\alpha^6 - A_9^i\alpha^5 + A_{10}^i\alpha^4 - A_{11}^i\alpha^3$$
$$+A_{12}^i\alpha^2 - A_{13}^i\alpha + A_{14}^i)(2\tilde{b}^2\tilde{\beta}) + (\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)(P_1\alpha^{12} - P_2\alpha^{11} + P_3\alpha^{10} - P_4\alpha^9 + P_5\alpha^8 - P_6\alpha^7 +$$
$$P_7\alpha^6 - P_8\alpha^5 + P_9\alpha^4 - P_{10}\alpha^3 + P_{11}\alpha^2 - P_{12}\alpha + P_{13}) = H_{00}^i(2\tilde{b}^2\tilde{\beta})(P_1\alpha^{12} - P_2\alpha^{11} + P_3\alpha^{10} - P_4\alpha^9$$
$$+P_5\alpha^8 - P_6\alpha^7 + P_7\alpha^6 - P_8\alpha^5 + P_9\alpha^4 - P_{10}\alpha^3 + P_{11}\alpha^2 - P_{12}\alpha + P_{13}). \quad (3.24)$$

Subtracting (3.24) from (3.23), we get

$$(A_1^i\alpha^{13} + A_3^i\alpha^{11} + A_5^i\alpha^9 + A_7^i\alpha^7 + A_9^i\alpha^5 + A_{11}^i\alpha^3 + A_{13}^i\alpha + A_{14}^i)(2\tilde{b}^2\tilde{\beta}) + (\tilde{A}^i\tilde{\alpha}^2 + \tilde{B}^i)$$
$$(P_2\alpha^{11} + P_4\alpha^9 + P_6\alpha^7 + P_8\alpha^5 + P_{10}\alpha^3 + P_{12}\alpha + P_{13}) = H_{00}^i(2\tilde{b}^2\tilde{\beta})(P_2\alpha^{11} + P_4\alpha^9 + P_6\alpha^7$$
$$+P_8\alpha^5 + P_{10}\alpha^3 + P_{12}\alpha + P_{13}) \quad (3.25)$$

Adding (3.23) and (3.24), we get

$$(A_2^i\alpha^{12}+A_4^i\alpha^{10}+A_6^i\alpha^8+A_8^i\alpha^6+A_{10}^i\alpha^4+A_{12}^i\alpha^2+A_{14}^i)(2\tilde{b}^2\tilde{\beta})+(\tilde{A}^i\tilde{\alpha}^2+\tilde{B}^i)$$
$$(P_1\alpha^{12}+P_3\alpha^{10}+P_5\alpha^8+P_7\alpha^6+P_9\alpha^4+P_{11}\alpha^2+P_{13})=H_{00}^i(2\tilde{b}^2\tilde{\beta})(P_1\alpha^{12}+P_3\alpha^{10}+P_5\alpha^8+$$
$$P_7\alpha^6+P_9\alpha^4+P_{11}\alpha^2+P_{13}). \tag{3.26}$$

From above equations, we observe that $\tilde{A}^i\tilde{\alpha}^2(P_1\alpha^{12}+P_3\alpha^{10}+P_5\alpha^8+P_7\alpha^6+P_9\alpha^4+P_{11}\alpha^2+P_{13})$ is divided by $\tilde{\beta}$. Since $\beta=\mu\tilde{\beta}$, then $\tilde{A}^i\tilde{\alpha}^2P_1\alpha^{12}$ can be divided by $\tilde{\beta}$. Since we have $\tilde{\beta}$ as prime with respect to α and $\tilde{\alpha}$, hence $\tilde{A}^i=\tilde{b}^2\tilde{s}_0^i-\tilde{b}^i\tilde{s}_0$, can be divided by $\tilde{\beta}$. Thus $\varphi^i(x)$ is a scalar function, providing

$$\tilde{b}^2\tilde{s}_0^i-\tilde{b}^i\tilde{s}_0=\tilde{\beta}\varphi^i. \tag{3.27}$$

Contracting (3.27) with $\tilde{y}_i:=\tilde{a}_{ij}y^j$, we get that $\varphi^i(x)=-\tilde{s}^i$. Then we have

$$s_{ij}=\frac{1}{b^2}(\tilde{b}_i\tilde{s}_j-\tilde{b}_j\tilde{s}_i)\ \text{provided}\ b^2\neq 0 \tag{3.28}$$

Thus by Lemma 3.2 $\tilde{F}=\dfrac{\tilde{\alpha}^2}{\tilde{\beta}}$ is a Douglas metric. As F and $\tilde{F}$ are having alike Douglas tensor, we consider them as Douglas metrics.

Hence proving proposition 3.2.

Now, we prove the next theorem which states,

Theorem 3.3. Let us consider an (α,β)-metric given by $F=\alpha+\beta+\dfrac{\beta^2}{\alpha}+\dfrac{\beta^3}{\alpha^2}$ and a Randers metric defined as $\tilde{F}=\dfrac{\tilde{\alpha}^2}{\tilde{\beta}}$ on an n-dimensional manifold with $n\geq 3$, where α and $\tilde{\alpha}$ denotes distinguish Riemannian metrics whereas β and $\tilde{\beta}$ denote non-zero 1-forms. The metrics F and $\tilde{F}$ are Projectively equivalent provided both are Douglas metrics and their geodesic coefficients are related following equations

$$G_\alpha^i=\tilde{G}_{\tilde{\alpha}}^i+\frac{1}{2\tilde{b}^2}(\tilde{\alpha}^2\tilde{s}^i+r_{00}\tilde{b}^i)+\theta y^i, \tag{3.29}$$

where $b^i:=a^{ij}b_j$, $\tilde{b}^i:=\tilde{a}^{ij}\tilde{b}_j$, $\tilde{b}^2:=\|\tilde{\beta}\|_\alpha^2$, $\tau=\tau(x)$ denotes a scalar function and $\theta=\theta_i y^i$ represents a 1-form on the manifold M.

Proof. On substituting (3.27) and (3.3) into (2.1), we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}[-\tilde{\alpha}^2\tilde{s}^i + (2\tilde{s}_0 y^i - r_{00}\tilde{b}^i) + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2} y^i]. \tag{3.30}$$

With the projective equivalence of F and $\tilde{F}$ we have a scalar function $P = P(x, y)$ on $TM \setminus \{0\}$ provided

$$G^i = \tilde{G}^i + Py^i, \tag{3.31}$$

From (3.17), (3.30) and (3.31), we have

$$[P - \frac{1}{\tilde{b}^2}(\tilde{s}_0 + \frac{2\tilde{r}_{00}\tilde{\beta}}{\tilde{\alpha}^2})]y^i = G^i_\alpha - \tilde{G}^i_{\tilde{\alpha}} - \frac{1}{2\tilde{b}^2}(\tilde{\alpha}^2\tilde{s}^i + r_{00}\tilde{b}^i). \tag{3.32}$$

Since RHS of (3.32) is quadratic in y, there exists a 1-form $\theta = \theta_i(x)y^i$ on M such that Thus we have

$$G^i_\alpha = \tilde{G}^i_{\tilde{\alpha}} + \frac{1}{2\tilde{b}^2}(\tilde{\alpha}^2\tilde{s}^i + r_{00}\tilde{b}^i) + \theta y^i, \tag{3.33}$$

Thus providing the necessity of the theorem.

Conversely, from (3.17), (3.31) and (3.14), we have

$$G^i = \tilde{G}^i + [\theta + \frac{1}{2\tilde{b}^2}(\tilde{\alpha}^2\tilde{s}^i + r_{00}\tilde{b}^i]y^i, \tag{3.34}$$

Hence F and $\tilde{F}$ are Projectively equivalent. Thus completing the proof of theorem 3.3.

4. METRICS WITH SPECIAL CURVATURE PROPERTIES

As is well known, the Berwald curvature tensor of a Finsler metric F is defined by

$$B := B^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^i,$$

where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$ and G^i are the spray coefficients of F. The mean Berwald curvature tensor is defined by

$$E := E_{ij} dx^i \otimes dx^j,$$

where $E_{ij} := \frac{1}{2} B^m_{mij}$ A Finsler metric is said to be of isotropic mean Berwald curvature if

$$E_{ij} := \frac{n+1}{2} c(x) F_{y^i y^j}$$

for some scalar function c(x) on M.

Clearly, the Finsler metric of isotropic Berwald curvature must be of isotropic mean Berwald curvature.

A Finsler metric F is said to have isotropic S-curvature if $S = n+1c(x)F$ for some scalar function c(x) on M.

Theorem 4.1. (See [5]) For a (α, β) -metric, the following are equivalent

(a) F has isotropic S-curvature, i.e. S = (n+1)c(x)F for some scalar function c(x) on M.
(b) F has isotropic mean Berwald curvature.
(c) β is a Killing one form of constant length with respect to α. This is equivalent to $r_{00} = s_0 = 0$..
(d) F has vanished S-curvature, i.e. S = 0.
(e) F is a weak Berwald metric, i.e. E = 0.

5. CONCLUSION

Therefore in the present paper a study has been done on the projective change between two important (α, β) -metrics, $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ (2nd approximate Matsumoto metric) and $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ (Randers metric) and also between 2nd approximate Matsumoto metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ (Kropina metric), where α and $\tilde{\alpha}$ denotes distinguish Riemannian metrics whereas β *and* $\tilde{\beta}$ represents two non-zero 1-forms, this study has provided two important examples under the topic Projective Change between two significant metrics.

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